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ON BURNSIDE'S LEMMA[1]

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ABSTRACT. Burnside's lemma on characters of finite groups is generalized, leading to the following theorem: if G is a simple group of order divisible by exactly 3 primes, and if one of the Sylow subgroups of G is cyclic, then for each Sylow subgroup P of G we have $C_G(P)=Z(P)$.

I. **Introduction.** Let G be a finite group and let X be an ordinary irreducible character of G of degree x. The basic step in the proof of Burnside's p^aq^b-theorem is the following:

LEMMA 1 [**2**, 18.1]. *Let C be a conjugate class of G and suppose that $(x, |C|)=1$. Then for $g\in C$ either $X(g)=0$ or $|X(g)|=x$.*

In this paper we apply Lemma 1, together with the following:

LEMMA 2. *Let $g\in G$. Then*:
(a) *If $X(g)=0$ and m is an integer prime to $o(g)$ then $X(g^m)=0$.*
(b) *If $X(g)=0$ and $o(g)=p$ a prime, then $p|x$.*

In order to prove:

THEOREM. *Let G be a simple group and suppose that $o(G)=p^ar^bq^c$, where p, r and q are primes. If a Sylow p-subgroup P of G is cyclic, then $C_G(H)=Z(H)$ for each Sylow subgroup H of G.*

II. **Generalizations of Burnside's lemma.** First we mention:

LEMMA 3. *Let S be a normal subset of G such that $X(s)=a$ for all $s\in S$, and suppose that $(x, |S|)=1$. Then either $a=0$ or $|a|=x$.*

PROOF. The proof follows exactly the lines of the proof of Lemma 1 in [**2**, 18.1]. The only thing to notice is that $|S|a/x$ is an algebraic integer, since that is the case for $|C|a/x$, where C is any of the conjugate classes contained in S.

In order to prove the Theorem we need the following

PROPOSITION. *Let $Z(G)=\{1\}$ and let H be a Hall π-subgroup of G.*

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Moreover, suppose that X is a faithful character of G of degree $x>1$. Then $x \mid |H|$ implies $C_G(H)=Z(H)$.

PROOF OF LEMMA 2 AND OF THE PROPOSITION. Since $(o(g), m)=1$, there exists an automorphism σ of $\langle g\rangle$ such that $g^\sigma=g^m$. It follows that $X(g^m)=X^\sigma(g)$ is an algebraic conjugate of $X(g)$, hence $X(g^m)=0$. If $o(g)=p$, then $X(g^i)=0$ for $i=1, \cdots, p-1$ and consequently $p \mid x$.

We proceed with a proof of the Proposition. It suffices to show that $C_G(H)$ is a π-subgroup of G. Suppose that $g\in C_G(H)$ is a π'-element of prime order p. Then $H\subset C_G(g)$ and consequently $(x, |C|)=1$, where C is the conjugate class of G containing g. It follows then by Lemma 1 that either $X(g)=0$ or $|X(g)|=x$. Since $p\nmid x$, $X(g)\neq 0$ by Lemma 2. If $|X(g)|=x$, then, by [2, 6.7], $g\in Z(G)$ in contradiction to our assumptions. Thus $C_G(H)$ is a π-group and $C_G(H)=Z(H)$.

III. **Proof of the Theorem.** In [1] Brauer has shown that $C_G(P)=P$. Let R and Q be an r-Sylow and a q-Sylow subgroup of G, respectively. It remains to be shown that $C_G(R)=Z(R)$ and $C_G(Q)=Z(Q)$.

Let $[N(P):P]=e$; then, since G is simple, $e>1$ and it follows by Proposition 2.1 and Corollary 2.1 of [3] that the principal p-block of G contains $e+1$ ordinary irreducible characters $X_0=1_G, X_1, \cdots, X_e$ of degrees $x_0=1, x_1, \cdots, x_e$ respectively, such that

$$1+\sum_{i=1}^{e}\epsilon_i x_i=0 \tag{1}$$

where $\epsilon_i=\pm 1$, $i=1, \cdots, e$ and $(x_i, p)=1$ for $i=1, \cdots, e$. Consequently $x_i=r^{b_i}q^{c_i}$ and by (1) there exist k and j, $1\leqq k, j\leqq e$, such that

$$x_j = r^{b_j} > 1, \qquad x_k = q^{c_k} > 1.$$

Our Proposition then implies that $C_G(R)=Z(R)$ and $C_G(Q)=Z(Q)$, thus completing the proof.

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