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AN INEQUALITY FOR COMPLEX LINEAR GROUPS OF SMALL DEGREE

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ABSTRACT. Let G be a finite complex irreducible linear group of degree less than $p-1$ for some fixed prime p, whose order is divisible by p to the first power only, and which has no normal Sylow p-subgroup. An inequality of Brauer, which bounds p by a function of the number of conjugate classes of p-elements, is improved.

The purpose of this note is to prove

THEOREM 1. *Let G be a finite group with the following properties: for some fixed prime p, a Sylow p-subgroup P of G has order p and is not normal in G; the number t of conjugate classes of p-elements of G is at least 3; and G has a faithful irreducible complex character χ of degree $d<p-1$. Then $p\leqq t^2-3t+1$.*

This improves Brauer's inequality $p\leqq t^3-t+1$ [2]. Hayden's result [6] that $t\geqq 6$ follows from Theorem 1 after the case $t=5$, $p=11$ is handled. All groups with $t\leqq 2$ and which satisfy the other hypotheses of Theorem 1 are known [8]. Apparently, no groups are known which satisfy these hypotheses with $6\leqq t<(p-1)/2$.

The proof is basically a series of observations on the methods of [5]. Note that [5, Theorem 1] shows that either t is even or $t=(p-1)/2$.

PROOF. Assume the hypotheses of Theorem 1. Then $d=p-(p-1)/t$ [2]. If $p=7$, no such groups exist [3], so we may assume $p>7$.

Let $e=(p-1)/t$. If $e\leqq 2$, then $t\geqq(p-1)/2$, and the theorem follows trivially for $p>7$. Assume henceforth $e>2$.

Let G_1 be the normal closure of P in G, and let $G_2=G_1'$. Feit's reduction argument [5, (6.1)] shows that $G_2=G_2'$, $G_2/\mathfrak{Z}(G_2)$ is simple, and $G_2/\mathfrak{Z}(G_2)\not\approx PSL_2(p)$. Also, G_2 and $\chi|_{G_2}$ satisfy the hypotheses of Theorem 1. Since d, and hence t, is the same for both groups, it suffices to prove the theorem in case $G=G_2$. Then by [5, (2.1)], G satisfies conditions (*) of [5] and $|\mathfrak{Z}(G)|\ |p-e$. Thus [5, Theorem 1] implies e is odd and t is even.

For the rest of this paper we use the following notation, in accordance with [5]: $N=\mathfrak{N}_G(P)$; $Z=\mathfrak{Z}(G)$; B_u is a p-block of defect 1

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corresponding to a linear character η^u of Z. The exceptional characters in B_u are denoted by $\chi_j^{(u)}$, $1 \leqq j \leqq t$. The sign $\delta_u = \pm 1$ as $\chi_j(1) \equiv \pm e \pmod{p}$. $\mathcal{O}$ is the ring of integers in a p-adic number field, $\mathcal{P}$ is the maximal ideal of $\mathcal{O}$, and $K = \mathcal{O}/\mathcal{P}$. If X is an $\mathcal{O}G$-module, $\overline{X} = X/\mathcal{P}X$. A typical indecomposable KN-module is denoted $V_{r,u}^{\lambda}$ where $1 \leqq r \leqq p$ and $1 \leqq u \leqq |Z|$. $\lambda = \alpha^h$ for some integer h, where α is the linear character of N given by $g^{-1}yg = y^{\alpha(g)}$, all $g \in N$, $y \in P$.

The following lemma is an obvious modification of [5, (3.8)]. It is proved by making the corresponding obvious changes in the proof of that result.

LEMMA 2. *Let group J satisfy* (*) *with $t \geqq 3$, and let M be an $\mathcal{O}$-free $\mathcal{O}J$-module which affords the character ξ. Suppose that $\xi|_Z = \xi(1)\eta^u$ where $\delta_u = 1$. Let $\xi = \alpha + \beta + \gamma$ where $\alpha = \sum_{j=1}^{t} h_j \chi_j^{(u)}$, β is a character in B_u which is orthogonal to every $\chi_j^{(u)}$, and γ is orthogonal to every character in B_u. Let $h = \sum_{j=1}^{t} h_j$. Then the following hold*:

(i) *If $\overline{M}$ is indecomposable and $\xi(1) \equiv e \pmod{p}$ or $\xi(1) \equiv e+1 \pmod{p}$ then $h \geqq 1$.*

(ii) *If $\overline{M} = W_1 \oplus W_2$ where each W_i is indecomposable and $\xi(1) \equiv 2e \pmod{p}$ then $h \geqq 2$.*

Now let χ, as in the statement of Theorem 1, be in the p-block B_u. χ is an exceptional character [5, §2]. Let X be an $\mathcal{O}$-free $\mathcal{O}G$-module which affords χ such that $\overline{X}$ is indecomposable. Then $\overline{X}|_N = V_{p-e,u}^{\mu}$ for some $\mu \in \langle \alpha \rangle$, and by [4, Lemma 3.7],

$$(\overline{X} \otimes \overline{X})|_N = \oplus \sum_{k=0}^{e-1} V_{2k+1,u+u}^{\mu^2\alpha^k} \oplus \sum_{k=e}^{p-e-1} V_{p,u+u}^{\mu^2\alpha^k}. \tag{3}$$

Since P is a T.I. set, for each $0 \leqq k \leqq e-1$ there is a unique indecomposable KG-module W_k so that $V_{2k+1,u+u}^{\mu^2\alpha^k}$ is the unique nonprojective indecomposable summand of $W_k|_N$. Then

$$\overline{X} \otimes \overline{X} = \oplus \sum_{k=0}^{e-1} W_k \oplus S,$$

where S is projective. There is a subset $\mathcal{S}_k$ of the integers j with $e \leqq j \leqq p-e-1$ such that, from (3),

$$W_k|_N = V_{2k+1,u+u}^{\mu^2\alpha^k} \oplus \sum_{j \in \mathcal{S}_k} V_{p,u+u}^{\mu^2\alpha^j}.$$

Assume $\mathcal{O}$ is large enough so that, by [5, (3.6)], for each k with $0 \leqq k < e$, there is an $\mathcal{O}$-free $\mathcal{O}G$-module M_k such that $\overline{M}_k \approx W_k \oplus W_{e-k-1}$, and so that there exists an $\mathcal{O}$-free $\mathcal{O}G$-module L with $\overline{L} \approx W_{(e-1)/2}$.

Let ζ be an exceptional character in B_{u+u}. By [5, (4.1)], $\delta_{u+u}=1$, so that $\zeta(1)\equiv e \pmod p$. Since $\zeta(1)\neq p-e$, we have $\zeta(1)>p$. Let Y be a modular constituent of ζ. Let $\dim_K Y=ap+y$, $0<y<p$. By an argument of Rothschild [7], $\sum y=e$, where the sum is taken over all modular constituents of ζ. It follows that there exists *some* constituent Y with $\dim_K Y>p$. Then $Y|_N$ contains at least one projective summand $V^{\gamma}_{p,u+u}$.

By Lemma 2, for each $0\leqq k<e$, the p-conjugates of ζ occur in the character afforded by M_k with a total multiplicity of at least 2. Thus Y occurs with multiplicity at least 2 as a constituent of $W_k\oplus W_{e-k-1}$. Similarly, Y occurs with multiplicity at least 1 as a constituent of $W_{(e-1)/2}$. Thus $V^{\gamma}_{p,u+u}$ occurs at least twice in a direct sum decomposition of $(W_k\oplus W_{e-k-1})|_N$, and at least once in a decomposition of $W_{(e-1)/2}|_N$. Hence $V^{\gamma}_{p,u+u}$ occurs at least $2(e-1)/2+1=e$ times in the decomposition (3). Since $|\langle\alpha\rangle|=e$, $V^{\gamma}_{p,u+u}$ can occur at most $t-1$ times, and at most $t-2$ times unless $\gamma=\mu^2\alpha^e=\mu^2$.

Suppose $\gamma=\mu^2$ and $e=t-1$. $\overline{X}\otimes\overline{X}$ is the direct sum of symmetric and skew summands, and by [1, Lemma 3.3], W_k and W_{e-k-1} are both symmetric summands for k odd, and both skew summands for k even, since e is odd. $V^{\gamma}_{p,u+u}$ occurs exactly twice as a summand of each $(W_k\oplus W_{e-k-1})|_N$, and exactly once in $W_{(e-1)/2}|_N$. It follows that $V^{\gamma}_{p,u+u}$ is a skew summand of $(\overline{X}\otimes\overline{X})|_N$ *more* times than it is a symmetric summand. However, [1, Lemma 3.3] also shows that $V^{\mu^2}_{p,u+u}$ appears $t/2$ times as a symmetric summand and $(t/2)-1$ times as a skew summand: contradiction.

Thus $e\leqq t-2$. Since e is odd and t even, we have $e\leqq t-3$. Since $e=(p-1)/t$, it follows that $p\leqq t^2-3t+1$, and Theorem 1 is proved.

Corollary 4. *Assume the hypotheses of Theorem* 1. *Then*

$$d \geqq p+(3/2)-(p+5/4)^{1/2}.$$

Proof. We know $e\leqq(p-1)/e-3$, so $e^2+3e-(p-1)\leqq 0$. Hence $e\leqq -3/2+(p+5/4)^{1/2}$. Since $d=p-e$, we are done.

References

1. H. I. Blau, *Under the degree of some finite linear groups*, Northern Illinois University, DeKalb, Ill., 1970 (preprint).

2. R. Brauer, *Some results on finite groups whose order contains a prime to the first power*, Nagoya Math. J. **27** (1966), 381–399. MR **33** #7402.

3. ———, *Über endliche lineare Gruppen von Primzahlgrad*, Math. Ann. **169** (1967), 73–96. MR **34** #5913.

4. W. Feit, *Groups with a cyclic Sylow subgroup*, Nagoya Math. J. **27** (1966), 571–584. MR **33** #7404.

5. ———, *On finite linear groups*, J. Algebra **5** (1967), 378–400. MR **34** #7632.

6. S. Hayden, *On finite linear groups whose order contains a prime larger than the degree*, Thesis, Harvard University, Cambridge, Mass., 1963.

7. B. Rothschild, *Degrees of irreducible modular characters of blocks with cyclic defect groups*, Bull. Amer. Math. Soc. **73** (1967), 102–104. MR **34** #4381.

8. H. F. Tuan, *On groups whose orders contain a prime number to the first power*, Ann. of Math. (2) **45** (1944), 110–140. MR **5**, 143.

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