

SEMILATTICE OF BISIMPLE REGULAR SEMIGROUPS

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ABSTRACT. The main purpose of this paper is to show that a regular semigroup S is a semilattice of bisimple semigroups if and only if it is a band of bisimple semigroups and that this holds if and only if \mathfrak{D} is a congruence on S . It is also shown that a quasiregular semigroup S which is a rectangular band of bisimple semigroups is itself bisimple.

In [3, Theorem 4.4] it was shown that a semigroup S is a semilattice of simple semigroups if and only if it is a union of simple semigroups. The purpose of this paper is to obtain corresponding results for a semigroup which is a semilattice of bisimple regular semigroups. Unfortunately, a semilattice of bisimple semigroups need not be a union of bisimple semigroups as illustrated by a simple ω -semigroup constructed by Munn [5]. However, we get some equivalent conditions for such semigroups. In particular we show that a regular semigroup is a semilattice of bisimple semigroups if and only if it is a band of bisimple semigroups.

1. Equivalent conditions. In this section we consider a set of equivalent conditions for a semigroup S to be a semilattice of bisimple semigroups. We adopt the terminology and notation of [2].

LEMMA 1.1. *Let S be a semilattice Ω of semigroups S_α and let D be a \mathfrak{D} -class of S . Then, either $S_\alpha \cap D = \square$ or $D \subseteq S_\alpha$.*

PROOF. Suppose $S_\alpha \cap D \neq \square$. Let $a_\alpha \in S_\alpha \cap D$. If $b_\beta \in S_\beta$ and $a_\alpha \mathfrak{D} b_\beta$, then there exists $c_\gamma \in S_\gamma$ ($\gamma \in \Omega$) such that $a_\alpha \mathfrak{R} c_\gamma$ and $c_\gamma \mathfrak{L} b_\beta$. Also $a_\alpha \mathfrak{R} c_\gamma$ implies that either $a_\alpha = c_\gamma$ in which case $\gamma = \alpha$, or there exist $x_\lambda \in S_\lambda$, $y_\mu \in S_\mu$ (λ, μ in Ω) such that $a_\alpha x_\lambda = c_\gamma$ and $c_\gamma y_\mu = a_\alpha$. However, since S is a semilattice of the semigroups S_α , $a_\alpha x_\lambda \in S_{\alpha\lambda}$ and $c_\gamma y_\mu \in S_{\gamma\mu}$. It follows that $\gamma = \alpha\lambda$ and $\alpha = \gamma\mu$ and so $\gamma \leq \alpha$ and $\alpha \leq \gamma$. Thus, in either case, $\gamma = \alpha$. Likewise, $\gamma = \beta$. Therefore, $\alpha = \beta$ and $D \subseteq S_\alpha$.

LEMMA 1.2. *Let S be a semigroup. If $ab\mathfrak{D}ba$ for all a, b in S , then \mathfrak{D} is a congruence on S .*

PROOF. Let $a\mathfrak{D}b$. Then there exists $x \in S$ such that $a\mathfrak{L}x$ and $x\mathfrak{R}b$. Since \mathfrak{L} is a right congruence and \mathfrak{R} is a left congruence, we have

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for $c \in S$, $ac \mathcal{L} xc$ and $cx \mathcal{R} cb$. That is, $ac \mathcal{D} xc$ and $cx \mathcal{D} cb$. Since by hypothesis $xc \mathcal{D} cx$, and $cb \mathcal{D} bc$, then $ac \mathcal{D} bc$ and also $ca \mathcal{D} cb$. Thus \mathcal{D} is a congruence on S .

LEMMA 1.3. *Let S be a semigroup which is a semilattice of bisimple semigroups. Then \mathcal{D} is a congruence on S .*

PROOF. Let $S = \bigcup_{\alpha \in \Omega} S_\alpha$ be a semilattice of the bisimple semigroups S_α . Then, by Lemma 1.1 each S_α is a union of \mathcal{D} -classes of S . However, since S_α is bisimple, any two elements of S_α are \mathcal{D} -related in S_α and hence in S . Consequently, S_α consists of a single \mathcal{D} -class of S . Thus the \mathcal{D} -classes of S are just the subsemigroups S_α and hence \mathcal{D} is a congruence on S .

As defined in [4], an element x in a semigroup S is said to be *quasiregular* if there exist elements $a, b, c, d \in S^1$ such that $x = xaxb = cxdx$. A semigroup S is *quasiregular* if every element of S is quasiregular. From the results in [4], we find that a regular semigroup is quasiregular, but the converse does not necessarily hold.

THEOREM 1.4. *A quasiregular semigroup S , which is a rectangular band of bisimple semigroups, is itself bisimple.*

PROOF. Let $S = \bigcup \{S_{i\lambda} \mid i \in I, \lambda \in \Lambda\}$ be a rectangular band of the bisimple semigroups $S_{i\lambda}$ and let S be quasiregular. Let $x_{i\lambda} \in S_{i\lambda}$, $y_{j\mu} \in S_{j\mu}$. Since $x_{i\lambda}y_{j\mu}$ is quasiregular, there exist elements a, b, c, d in S^1 such that

$$(1.4.1) \quad x_{i\lambda}y_{j\mu}ax_{i\lambda}y_{j\mu}b = x_{i\lambda}y_{j\mu}$$

and

$$(1.4.2) \quad cx_{i\lambda}y_{j\mu}dx_{i\lambda}y_{j\mu} = x_{i\lambda}y_{j\mu}.$$

However, (1.4.1) implies that $x_{i\lambda}y_{j\mu}ax_{i\lambda} \mathcal{R} x_{i\lambda}y_{j\mu}$ and (1.4.2) implies that $x_{i\lambda}y_{j\mu} \mathcal{L} y_{j\mu}dx_{i\lambda}y_{j\mu}$, and hence $x_{i\lambda}y_{j\mu}ax_{i\lambda} \mathcal{D} y_{j\mu}dx_{i\lambda}y_{j\mu}$. Since $x_{i\lambda}y_{j\mu}ax_{i\lambda} \in S_{i\lambda}$ and $y_{j\mu}dx_{i\lambda}y_{j\mu} \in S_{j\mu}$, and $S_{i\lambda}$ and $S_{j\mu}$ are bisimple, it follows that $S_{i\lambda}$ and $S_{j\mu}$ are contained in a single \mathcal{D} -class of S for all $i, j \in I$ and $\lambda, \mu \in \Lambda$. Thus S is bisimple.

THEOREM 1.5. *The following are equivalent for any regular semigroup S .*

- (A) S is a semilattice of bisimple semigroups.
- (B) For $a, b \in S$, $ab \mathcal{D} ba$.
- (C) \mathcal{D} is a congruence on S .
- (D) If $e, f, g \in E(S)$, where $E(S)$ is the set of idempotents of S , then $e \mathcal{D} f \Rightarrow$ (i) $eg \mathcal{D} fg$ and (ii) $ge \mathcal{D} gf$.
- (E) S is a band of bisimple semigroups.

PROOF. Assume (A) and let S be a semilattice Ω of bisimple semigroups S_α . Let $a, b \in S$, and denote the \mathfrak{D} -classes containing a and b by D_a and D_b respectively. Then, by Lemma 1.1, $D_a \subseteq S_\alpha$ and $D_b \subseteq S_\beta$ for some α, β in Ω . Hence $ab \in D_a D_b \subseteq S_\alpha S_\beta \subseteq S_{\alpha\beta}$ and $ba \in D_b D_a \subseteq S_\beta S_\alpha \subseteq S_{\alpha\beta}$. Since $S_{\alpha\beta}$ is bisimple, we conclude that $ab \mathfrak{D} ba$. Thus (A) implies (B). (B) implies (C) by Lemma 1.2. Also (C) implies (D) trivially. We shall now show that (D) implies (C). Suppose $a \mathfrak{D} b$ and $c \in S$. Let a', b', c' be one of the inverses of a, b, c respectively. Since $a'a \mathfrak{L} a$ and \mathfrak{L} is a right congruence, we have $a'a \cdot cc' \mathfrak{L} a \cdot cc'$. Also, since $cc' \mathfrak{R} c$ and since \mathfrak{R} is a left congruence, we have $a \cdot cc' \mathfrak{R} ac$. Thus $a' acc' \mathfrak{D} ac$ and likewise $b' bcc' \mathfrak{D} bc$. Since $a' a \mathfrak{D} b' b$, it follows from (D) that $a' acc' \mathfrak{D} b' bcc'$ and hence $ac \mathfrak{D} bc$. In a similar manner, $ca \mathfrak{D} cb$ and thus \mathfrak{D} is a congruence.

To show that (C) implies (E), suppose $a \mathfrak{D} b$ so that $a \mathfrak{D} bb'$ where b' is an inverse of b . Since \mathfrak{D} is a congruence, this implies that $ab \mathfrak{D} bb'b$ or $ab \mathfrak{D} b$. Thus each \mathfrak{D} -class of S is a subsemigroup of S and moreover, since S is regular, it is a bisimple semigroup [2, p. 61, Example 6]. Denoting the \mathfrak{D} -classes of S by S_α , $\alpha \in \Omega$, we have $S = \bigcup \{S_\alpha \mid \alpha \in \Omega\}$. Further, since \mathfrak{D} is a congruence, $S_\alpha S_\beta$ is contained in a semigroup S_γ for some $\gamma \in \Omega$. If we define $\alpha\beta = \gamma$, then, noting that $S_\alpha S_\alpha \subseteq S_\alpha$, Ω becomes a band and thus S is a band of bisimple semigroups.

Finally, in order to show that (E) implies (A), we use the following result by Clifford [1]. "Let \mathfrak{C} be a class of semigroups. If a semigroup S is a band of semigroups of type \mathfrak{C} , then S is a semilattice of semigroups each of which is a rectangular band of semigroups of type \mathfrak{C} ." Now assuming that S is a band of bisimple semigroups, S is a semilattice Ω of semigroups S_α ($\alpha \in \Omega$), where each S_α is a rectangular band of bisimple semigroups. If $a_\alpha \in S_\alpha \subseteq S$, since S is regular, there exists $a'_\alpha \in S$ such that $a_\alpha a'_\alpha a_\alpha = a_\alpha$ and further $a_\alpha \mathfrak{D} a'_\alpha$. However, by Lemma 1.1 each S_α is a union of \mathfrak{D} -classes of S and hence $a'_\alpha \in S_\alpha$. Thus S_α is regular, and in particular, quasiregular. Consequently, by Theorem 1.4, S_α is bisimple, and we have (A). This completes the theorem.

2. Examples. In this section we consider some examples to show that Theorem 1.5 is not true in general for semigroups which are not regular. First we need a lemma.

LEMMA 2.1. *Let S be a Baer-Levi semigroup [3, p. 82] and T be any semigroup. Let $A = T \times S$ be the direct product of T and S . Then $(t, s) \mathfrak{D} (t', s')$ if and only if $t \mathfrak{R} t'$, in T , and consequently the \mathfrak{D} -classes of A are just the sets $R \times S$ where R is an \mathfrak{R} -class of T .*

PROOF. The proof of the "only if" part follows if we show that the \mathfrak{L} -classes of A contain single elements. Now suppose $(t, s)\mathfrak{L}(t', s')$. Then either

(i) $(t, s) = (t', s')$, or

(ii) there exists x, y in T and a, b in S such that $(x, a)(t, s) = (t', s')$ and $(y, b)(t', s') = (t, s)$.

If (ii) holds, we have $as = s'$ and $bs' = s$. Thus, $(ba)s = b(as) = bs' = s$ which contradicts [3, Lemma 8.3]. Hence (ii) cannot hold, and we have established the assertion.

The "if" part is clear, since S is right simple and any two elements of S are \mathfrak{R} -related.

EXAMPLE 1. Let $E_1 = \{e_1, e_2\}$ be a right zero semigroup of order 2, $E_2 = \{e_3, e_4, e_5\}$ be a left zero semigroup of order 3, and let $E = E_1 \cup E_2$ be a disjoint union where

$$e_i e_j = e_j e_i = e_j \quad \text{for } i = 1, 2, j = 3, 4, \\ e_1 e_5 = e_3, \quad e_5 e_1 = e_5, \quad e_2 e_5 = e_4, \quad e_5 e_2 = e_5.$$

The fact that E is a semigroup can be easily verified. Moreover, $e_1 \mathfrak{R} e_2$ and $e_3 \mathfrak{L} e_4 \mathfrak{L} e_5$. Set $T = E \times S$, the direct product of E and S , where S is the Baer-Levi semigroup. Using Lemma 2.1, the \mathfrak{D} -classes of T are given by

$$D_1 = \{e_1, e_2\} \times S, \quad D_i = \{e_{i+1}\} \times S \quad (i = 2, 3, 4).$$

Clearly, T is a band of the bisimple semigroups $S_i = \{e_i\} \times S$, $i = 1, 2, 3, 4, 5$. However, if $s \in S$, then $(e_1, s)\mathfrak{D}(e_2, s)$. But $(e_1, s)(e_5, s) = (e_3, s^2) \in D_2$ and $(e_2, s)(e_5, s) = (e_4, s^2) \in D_3$. Thus \mathfrak{D} is not a congruence. Moreover, T is not a semilattice of bisimple semigroups, for otherwise, Lemma 1.3 would imply that \mathfrak{D} is a congruence. Further T does not satisfy the condition (B) of Theorem 1.5, since in that case Lemma 1.2 would again imply that \mathfrak{D} is a congruence.

EXAMPLE 2. Let $S = \{0, a, 1\}$ where $a^2 = 0$, with 0 as the zero element and 1 as the identity element. Then \mathfrak{D} is a congruence, being the identity relation. However, S is neither a band of bisimple semigroups nor a semilattice of bisimple semigroups.

EXAMPLE 3. Let $E = \{e_1, e_2\}$ be a left zero semigroup and let S be the Baer-Levi semigroup. Set $T = E \times S$. Then T is a semigroup and by Lemma 2.1, the \mathfrak{D} -classes of T are:

$$D_1 = \{e_1\} \times S \quad \text{and} \quad D_2 = \{e_2\} \times S.$$

Moreover, D_1 and D_2 are bisimple semigroups such that $D_1 D_2 \subseteq D_1$ and $D_2 D_1 \subseteq D_2$. Thus \mathfrak{D} is a congruence on T and T is a band of bi-

simple semigroups. However, if $a = (e_1, s)$ and $b = (e_2, s)$ for $s \in S$, then $ab \in D_1$ and $ba \in D_2$. Hence, ab and ba are not \mathfrak{D} -related. This will further imply that T is not a semilattice of bisimple semigroups.

REMARK. If, in Example 3 above, we write $S_{11} = D_1$, $S_{21} = D_2$, we find that S is a rectangular band of the bisimple semigroups S_{11} and S_{21} . But S is not bisimple. This shows that Theorem 1.4 is not true in general for semigroups which are not quasiregular.

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