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SIMPLE ZEROS OF SOLUTIONS OF nTH-ORDER LINEAR DIFFERENTIAL EQUATIONS[1]

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ABSTRACT. Let the nth-order linear differential equation $Ly=0$ have a nontrivial solution with n zeros (counting multiplicities) on an interval $[\alpha, \beta]$. A condition under which $Ly=0$ has a solution with n simple zeros on $[\alpha, \beta]$ is established.

Also, a new proof is given for a known result concerning an interval of the type $[\alpha, \beta)$.

Let the differential equation

$$y^{(n)} + p_{n-1}y^{(n-1)} + \cdots + p_0 y = 0, \tag{1}$$

where $p_0, p_1, \cdots, p_{n-1}$ are real-valued and continuous on an interval I, have a nontrivial solution which has n zeros (counting multiplicities) on I. In this paper we shall be concerned with the following question: Does equation (1) have a nontrivial solution with n distinct zeros on I?

Hartman [1] proved that equation (1) has a nontrivial solution with n zeros (counting multiplicities) on (α, β) if and only if there is a nontrivial solution with n distinct zeros on (α, β). A similar result was obtained by Opial [3], under the condition that $p_0, p_1, \cdots, p_{n-1}$ be summable on (α, β). In a recent paper, Sherman [6] established the following theorem:

THEOREM 1 [6]. *Suppose there is a nontrivial solution of* (1) *with a zero at* α *and* n *zeros on* $[\alpha, \beta)$. *Then there is a solution with a simple zero at* α *whose first* n *zeros on* $[\alpha, \beta)$ *are simple zeros. The interval* $[\alpha, \beta)$ *cannot be replaced by the closed interval* $[\alpha, \beta]$.

In view of the interesting and useful nature of this theorem, the presentation of different proofs appears warranted. We shall provide an alternative, shorter proof of Theorem 1. This alternative method of proof has an added advantage; it sheds light on how Theorem 1 may be modified so as to hold for the closed interval $[\alpha, \beta]$. In fact, we shall prove the following statements.

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THEOREM 2. *Suppose equation* (1) *has a nontrivial solution with a zero at* α, n *zeros on* $[\alpha, \beta]$, *and* $n-2$ *zeros on* (α, β). *Then there exists a solution of* (1) *with a simple zero at* α, *such that its first* n *zeros on* $[\alpha, \beta]$ *are simple. The number of zeros on* (α, β), $n-2$, *cannot in general be replaced by a smaller number.*

We require a few definitions before proceeding with proofs. The first conjugate point $\eta_1(\alpha)$ of a point $\alpha \in I$ is the smallest number $\beta > \alpha$, $\beta \in I$, such that there exists a nontrivial solution of (1) which vanishes at α and has n zeros on $[\alpha, \beta]$. A nontrivial solution of (1) which has n zeros on $[\alpha, \eta_1(\alpha)]$ is called an extremal solution for the interval $[\alpha, \eta_1(\alpha)]$. A nontrivial solution of (1) is said to have an $i_1 - i_2 - \cdots - i_m$ distribution of zeros on I if it has a zero of order at least i_k at $x_k \in I$, $i = 1, 2, \cdots, m$, $x_1 < x_2 < \cdots < x_m$.

Let $y_1, y_2, \cdots, y_n$ be n linearly independent solutions of (1). Define

$$w(x; x_1^{[k_1]}, x_2^{[k_2]}, \cdots, x_m^{[k_m]})$$

$$(2) \qquad = \begin{vmatrix} y_1(x) & y_2(x) & \cdots y_n(x) \\ y_1(x_1) & y_2(x_1) & \cdots y_n(x_1) \\ y_1'(x_1) & y_2'(x_1) & \cdots y_n'(x_1) \\ & \cdot\ \cdot\ \cdot\ \cdot\ \cdot & \\ y_1^{(k_1-1)}(x_1) & y_2^{(k_1-1)}(x_1) & \cdots y_n^{(k_1-1)}(x_1) \\ y_1(x_2) & y_2(x_2) & \cdots y_n(x_2) \\ & \cdot\ \cdot\ \cdot\ \cdot\ \cdot & \\ y_1(x_m) & y_2(x_m) & \cdots y_n(x_m) \\ & \cdot\ \cdot\ \cdot\ \cdot\ \cdot & \\ y_1^{(k_m-1)}(x_m) & y_2^{(k_m-1)}(x_m) & \cdots y_n^{(k_m-1)}(x_m) \end{vmatrix},$$

$1 \leqq m \leqq n-1$, $k_1 + k_2 + \cdots + k_m = n-1$; and put

$$w(x; x_1^{[1]}, x_2^{[1]}, \cdots, x_{n-1}^{[1]}) = w(x; x_1, x_2, \cdots, x_{n-1}).$$

For fixed x_i, $i = 1, 2, \cdots, m$, $w(x) = w(x; x_1^{[k_1]}, \cdots, x_m^{[k_m]})$ is a solution of (1) with a zero of order k_i at x_i, $i = 1, 2, \cdots, m$. Moreover, it is a continuous function of the terms, e.g., $y_2^{(k_1-1)}(x_1)$, appearing in the determinant.

PROOF OF THEOREM 1. Let $\mathcal{Y}$ be an extremal solution for $[\alpha, \eta_1(\alpha)] \subset [\alpha, \beta)$ which has the largest number p of distinct zeros $x_1, x_2, \cdots, x_p$ on $(\alpha, \eta_1(\alpha))$ among all the extremal solutions for $[\alpha, \eta_1(\alpha)]$. Set $\alpha = x_0$ and $\eta_1(\alpha) = x_{p+1}$. Suppose $\mathcal{Y}$ has a zero of order k_i at x_i, $i = 1, 2,$

$\cdots, p+1$. We shall assume in addition that $\mathcal{Y}$ has the highest order k_{p+1} of zero at $\eta_1(\alpha)$ among all the extremal solutions which vanish at $x_0, x_1, \cdots, x_{p+1}$. If $p=n-2$, there is nothing to prove; $\mathcal{Y}$ has simple zeros at x_i, $i=0, 1, \cdots, p+1$ [5, Theorem 7], and has no other zeros on $[\alpha, \eta_1(\alpha)]$. Assume $p<n-2$. We first prove the theorem under the further assumption $k_{p+1}>1$. The function w_1 defined by

$$w_1(x) = w(x; x_0^{[k_0]}, x_1^{[k_1]}, \cdots, x_p^{[k_p]}, x_{p+1}^{[k_{p+1}-1]}),$$

where $k_0=n-k_1-\cdots-k_p-k_{p+1}$, is a nontrivial solution of (1) with zeros of order k_i at x_i, $i=0, 1, \cdots, p+1$. It is nontrivial; for it would otherwise imply the existence of a nontrivial solution with $p+1$ distinct zeros on $(\alpha, \eta_1(\alpha))$, contrary to the choice of p. It will be shown that the multiple zeros of w_1 can be separated into simple zeros in a continuous manner. Define w_2 by

$$w_2(x) = \frac{\prod_{l=0}^{k_{p+1}-2} l! \left[\prod_{i=0}^{p}\prod_{j=0}^{k_i-1} j!\right]}{\prod_{l=0}^{k_{p+1}-2} (\zeta_{p+1\,l} - x_{p+1})^l \left[\prod_{i=0}^{p}\prod_{j=0}^{k_i-1} (\zeta_{ij} - x_i)^j\right]}$$

$$\cdot w(x; x_0, \zeta_{01}, \zeta_{02}, \cdots, \zeta_{0\,k_0-1}, x_1, \zeta_{11}, \cdots, x_p,$$

$$\zeta_{p1}, \cdots, \zeta_{p\,k_p-1}, \zeta_{p+1\,1}, \cdots, \zeta_{p+1\,k_{p+1}-2}, x_{p+1} - \delta_1),$$

$$x_0<\zeta_{01}<\zeta_{02}< \cdots <x_p<\zeta_{p1}< \cdots <\zeta_{p\,k_p-1}<\zeta_{p+1\,1}< \cdots <x_{p+1}-\delta_1.$$

For a given $\epsilon>0$, there exist $\delta_1>0$ and $\delta>0$ such that w_2 is not identically zero and

$$(3) \qquad |w_1(x) - w_2(x)| < \epsilon, \qquad x \in [\alpha, \gamma], \quad \eta_1(\alpha) < \gamma < \beta,$$

if $|\zeta_{ij}-x_i| <\delta$ for $i=0, 1, \cdots, p+1, j=1, 2, \cdots, k_i-1$, and for $i=p+1, j=1, 2, \cdots, k_{p+1}-2$. This follows from the Taylor formula which may be written as

$$y_k^{(j)}(x_i)$$
$$=j!\left[\frac{y_k(\zeta_{ij})-y_k(x_i)}{(\zeta_{ij}-x_i)^j} - \frac{y_k'(x_i)}{(\zeta_{ij}-x_i)^{j-1}} - \cdots - \frac{y_k^{(j-1)}(x_i)}{(j-1)!(\zeta_{ij}-x_i)}\right] + \epsilon_{ijk},$$

where $\epsilon_{ijk}\to 0$ as $\zeta_{ij}\to x_i$, $i=0, 1, \cdots, p+1, j=1, 2, \cdots, k_i-1, k=1, 2, \cdots, n$. When these formulas are substituted in w_1 and the continuity of w_1 with respect to the elements in the determinant—which defines w_1—is noted, the assertion follows. Evidently, w_2 is a solution of (1) with simple zeros at $x_0, \zeta_{01}, \zeta_{02}, \cdots, x_p, \zeta_{p1}, \cdots,$

$x_{p+1}-\delta_1$. Furthermore, for a sufficiently small ϵ, w_2 has an nth simple zero in a given arbitrarily small interval $[\eta_1(\alpha), \eta_1(\alpha)+\epsilon_1]$, $\epsilon_1>0$. This is due to (3) and the maximality of $\mathcal{Y}$ with respect to the order of the zero at $\eta_1(\alpha)$, since the zeros can disappear only in pairs. This establishes the theorem for $k_{p+1}>1$.

If $k_{p+1}=1$, we define w_1 by

$$w_1(x) = w(x; x_0^{[k_0]}, x_1^{[k_1]}, \cdots, x_p^{[k_p]}),$$

where $k_0=n-1-k_1-\cdots-k_p$. The rest of the proof is similar to the case $k_{p+1}>1$.

The interval $[\alpha, \beta)$ cannot be replaced by the closed interval $[\alpha, \beta]$. This is easily confirmed by examining the extremal solutions of the equation

$$y^{(\mathrm{iv})} - y = 0. \tag{4}$$

This equation has an oscillatory solution, sin x, which vanishes at $x=0$. Thus $\eta_1(0)<\infty$. According to a note made in [**2**], every extremal solution of (4) has zeros of order exactly 2 at $x=0$ and $x=\eta_1(0)$ and does not vanish on $(0, \eta_1(0))$. Therefore, no solution of (4) has four simple zeros on $[0, \eta_1(0)]$.

Proof of Theorem 2. If $\eta_1(\alpha)<\beta$, there is a γ, $\eta_1(\alpha)<\gamma<\beta$, such that (1) has a solution with a zero at α and n zeros on $[\alpha, \gamma)$. Thus, by Theorem 1, equation (1) has a solution with a simple zero at α whose first n zeros on $[\alpha, \gamma)\subset[\alpha, \beta]$ are simple. Note that in this case the existence of $n-2$ zeros on (α, β) was not used.

If $\eta_1(\alpha)=\beta$, there exists an extremal solution for $[\alpha, \eta_1(\alpha)]$ with n distinct zeros on $[\alpha, \eta_1(\alpha)]$, by Theorem 2.2 in [**2**]. Of course, these n distinct zeros must be simple.

It remains to show that the number of zeros on (α, β), $n-2$, cannot be replaced by a smaller number. We shall prove this by exhibiting a fourth-order equation which has an extremal solution with a 1-1-2 distribution of zeros, but which does not have an extremal solution with a 1-1-1-1 distribution of zeros. The equation

$$(6x^2 - 8x + 3)y^{(\mathrm{iv})} - (12x - 8)y''' + 12y'' = 0 \tag{5}$$

has linearly independent solutions $y_1=x$, $y_2=-x(1-x)^2$, $y_3=x(1-x)^3$ and $y_4=1$. Hence, for $0<x<1$,

$$y_1 = x > 0, \qquad \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = 2x^2(1-x) > 0,$$

and

$$\begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix} = 6x^3(1 - x)^2 > 0.$$

According to a result of Pólya [**4**, Theorem II], these inequalities imply the disconjugacy of equation (5) on (0, 1). Moreover, this in turn implies the disconjugacy of (5) on [0, 1) [**6**, Theorem 5]. Thus, for example, y_3 is an extremal solution for [0, 1] with a 1-3 distribution of zeros. Likewise, $y_5 = x(\lambda - x)(1-x)^2$, $0<\lambda<1$, is an extremal solution with a 1-1-2 distribution of zeros. However, it is easily confirmed that no extremal solution of (5) can have a 3-1 distribution of zeros on [0, 1]. Hence, no solution of (5) can have four simple zeros on [0, 1] (cf. [**2**, Theorem 2.2]).

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