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A THEOREM ON PERFECT MAPS

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1. **Introduction.** The purpose of this note is to give a short proof of the following theorem, and to indicate some applications.

THEOREM 1.1. *If $f:X\rightarrow Y$ is perfect,*[2] *and $g:X\rightarrow Z$ is continuous with Z Hausdorff, then $(f, g):X\rightarrow Y\times Z$ is perfect.*[3]

Theorem 1.1 is implicit in the proofs of two results of A. V. Arhangel'skiĭ [**1**, Lemmas 1 and 3],[4] and also follows immediately from a result on set-valued maps which is stated by Z. Frolík in [**3**, Proposition 6 and remark at end of §1]. We prove Theorem 1.1 in §2.

The following is a direct consequence of Theorem 1.1.

COROLLARY 1.2. *If X admits a perfect map into a topological space Y, and a continuous one-to-one map into a Hausdorff space Z, then X is homeomorphic to a closed subspace of $Y\times Z$.*

Corollary 1.2 immediately implies the nontrivial part ((a)→(b)) of the following result, which was essentially obtained by J. Nagata in [**4**, Theorem 1], and which also follows from J. van der Slot [**5**, Theorem, p. 21].[5]

COROLLARY 1.3. If *Y is any topological space, then the following properties of a completely regular space X are equivalent.*

(a) *There exists a perfect map $f:X\rightarrow Y$.*

(b) *X is homeomorphic to a closed subspace of $Y\times Z$ for some compact Hausdorff space Z.*

(c) *X is homeomorphic to a closed subspace of $Y\times Z$ for some compact space Z.*

In a different direction, the following result of Bourbaki [**2**, p. 115,

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[2] A map $f:X\rightarrow Y$ (not necessarily onto) is *perfect* if f is closed (i.e. $f(A)$ is closed in Y for every closed $A\subset X$) and $f^{-1}(y)$ is compact for every $y\in Y$. (Perfect maps are called *proper* by Bourbaki [**2**].)

[3] We define $(f, g)(x)=(f(x), g(x))$.

[4] It appears that Arhangel'skiĭ calls a map $f:X\rightarrow Y$ perfect in [**1**] if the map $f:X\rightarrow f(X)$ is perfect in our terminology. Thus Arhangel'skiĭ does not require $f(X)$ to be closed in Y.

[5] I am grateful to A. V. Arhangel'skiĭ for this reference.

Proposition 5(d)] is also an easy consequence of Theorem 1.1, as our proof in §3 will show.[6]

COROLLARY 1.4. *Let $\alpha:A\to B$ and $\beta:B\to C$ be continuous, and suppose that $\beta\circ\alpha$ is perfect and that B is Hausdorff. Then α is perfect.*

In conclusion, let us observe that the following useful known result follows immediately from Corollary 1.4 (by taking $\alpha:A\to B$ to be the injection map).

COROLLARY 1.5. *If $\gamma:A\to C$ is perfect, and if γ has a continuous extension $\beta:B\to C$ for some Hausdorff space $B\supset A$, then A is closed in B.*

2. **Proof of Theorem 1.1.** Clearly (f, g) is the composition of the following two maps:

$$X \xrightarrow{(i_X,\, g)} X\times Z \xrightarrow{f\times i_Z} Y\times Z.$$

Now (i_X, g) maps X homeomorphically onto the graph of g, which is closed in $X\times Z$ because Z is Hausdorff. Since $f\times i_Z$ is the product of two perfect maps, it is perfect by [2, p. 114, Proposition 4]. Hence (f, g) is perfect.[7]

3. **Proof of Corollary 1.4.** If $\gamma=(\alpha,\ \beta\circ\alpha)$, then $\gamma:A\to B\times C$ is perfect by Theorem 1.1. Now the projection $\pi:B\times C\to B$ maps the graph G_β of β homeomorphically onto B. Since $\gamma(A)\subset G_\beta$ and $\alpha=(\pi|G_\beta)\circ\gamma$, it follows that α is perfect.

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[6] As a partial converse, Corollary 1.4 implies the slight weakening of Theorem 1.1 which results from assuming that Y (as well as Z) is Hausdorff.

[7] The assumption that Z is Hausdorff cannot be dropped, or even weakened to T_1. Example: $X=Y=$interval I with usual topology, $Z=I$ with cofinite topology, $f=g=i_X$.