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# INTEGRAL RING EXTENSIONS AND PRIME IDEALS OF INFINITE RANK

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ABSTRACT. An example is constructed showing that for an integral ring extension $R \subset T$, and a prime ideal $P$ of $R$ having infinite rank, it can happen that in $T$ each prime ideal lying over $P$ has finite rank.

By the rank (or height) of a prime ideal $P$ in a commutative ring $R$ is meant the maximal length of descending chains of prime ideals of $R$ starting with $P$. Thus $P$ has rank $n$ if there exists a descending chain $P = P_0 \supset P_1 \supset \cdots \supset P_n$, but no such chain of longer length; and $P$ has infinite rank (or rank $\infty$) if there exist arbitrarily long chains of primes descending from $P$. Let $R \subset T$ be a pair of commutative rings (having a common identity). One says that the going up property (GU) holds for the pair $R \subset T$ if whenever $P \subset P_0$ are prime ideals in $R$ and $Q$ is a prime of $T$ such that $Q \cap R = P$, then there exists a prime $Q_0$ in $T$ such that $Q \subset Q_0$ and $Q_0 \cap R = P_0$. It is well known that if $T$ is integral over $R$, then GU holds for the pair $R \subset T$; and it can be readily seen that if $R \subset T$ satisfies GU and $P$ is a prime ideal in $R$ of rank $n$, then there exists in $T$ a prime ideal $Q$ such that $Q$ has rank $\geqq n$ and $Q \cap R = P$ [**3**, Theorem 46, p. 31]. We show, however, that this result cannot be extended to primes of rank $\infty$ even for $R$ an integral domain and $T$ the integral closure of $R$. Of course GU insures that there can be no fixed bound on the ranks of the primes of $T$ lying over a rank $\infty$ prime $P$ of $R$. Thus in our example there must be infinitely many primes of $T$ lying over $P$. In particular, $T$ cannot be a finite $R$-module [**1**, p. 40].

The idea involved in our construction may be stated as follows.

LEMMA. *Let $R$ be a quasi-local domain with maximal ideal $P$ and quotient field $K$. Assume that for each positive integer $n$ there exists a valuation ring of $K$ containing $R$ and having rank $n$, but that $R$ is contained in no valuation ring of $K$ having infinite rank. Let $T$ be the integral closure of $R$. If $T$ is a Prüfer domain, then $P$ has infinite rank but each prime ideal of $T$ has finite rank.*

PROOF. If $Q$ is a prime ideal of $T$, then the localization $T_Q$ is a

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valuation ring and the rank of the prime ideal $Q$ equals the rank of the valuation ring $T_Q$. Thus each prime of $T$ has finite rank and by intersecting chains of primes of $T$ with $R$, we see that $P$ has infinite rank.

CONSTRUCTION OF THE EXAMPLE. Let $k$ be an arbitrary field and let $\{x_i\}_{i=1}^{\infty}$ be a collection of indeterminates over $k$. We construct a rank one valuation ring $V_1$ on the field $K=k(x_1, x_2, \cdots)$ such that $V_1$ has the form $k+M_1$ where $M_1$ is the maximal ideal of $V_1$. This can be done, for example, by mapping the $x_i$ onto rationally independent real numbers and then extending this map to a valuation of $K$ trivial on $k$. The $x_i$ having rationally independent values assures that $k$ maps isomorphically onto the residue field of $V_1$ and hence that $V_1=k+M_1$. For each integer $n \geqq 2$, let $L_n$ denote the field $k(\{x_i \mid i \leqq n \text{ or } i \geqq 2n\})$. Thus $K=L_n(x_{n+1}, \cdots, x_{2n-1})$ and $x_{n+1}, \cdots, x_{2n-1}$ are algebraically independent over $L_n$. Consider the valuation ring $V_1 \cap L_n$. By mapping $x_{n+1}, \cdots, x_{2n-1}$ onto suitably chosen elements of a suitable totally ordered abelian group containing the value group of $V_n$ we can obtain a valuation ring $V_n$ of $K$ such that:

(1) $V_n \cap L_n = V_1 \cap L_n$.

(2) $V_n$ has rank $n$.

(3) $V_n$ has the form $k+M_n$ where $M_n$ is the maximal ideal of $V_n$.

See, for example, [1, Proposition 1, p. 161].

Let $P=\bigcap_{i=1}^{\infty} M_i$ and let $R=k+P$. We note that $R$ is a quasi-local domain with maximal ideal $P$. For if $\alpha$ is a nonzero element of $k$ and $m \in P$, then $(\alpha+m)^{-1}=\alpha^{-1}+m'$, where $m'=-m/\alpha(\alpha+m) \in M_i$ for each $i$, so $m' \in P$. Let $T$ be the integral closure of $R$.

CLAIM. $T$ is a Prüfer domain with quotient field $K$, $T=\bigcap_{i=1}^{\infty} V_i$, and no valuation ring between $T$ and $K$ has infinite rank.

PROOF. Let $K_n=k(x_1, \cdots, x_n)$, $R_n=R \cap K_n$, and let $T_n$ be the integral closure of $R_n$. Note that for $s \geqq n$, $V_s \cap K_n = V_1 \cap K_n$. Hence

$$R_n = k + \left( \bigcap_{i=1}^{n-1} M_i \cap K_n \right).$$

We show that $T_n = \bigcap_{i=1}^{n-1} V_i \cap K_n$. If $y \in \bigcap_{i=1}^{n-1} V_i \cap K_n$ then there exists $a_i \in k$ such that $y-a_i \in M_i$, for each $i$ such that $1 \leqq i < n$. It follows that $\prod_{i=1}^{n-1} (y-a_i) \in \bigcap_{i=1}^{n-1} M_i \cap K_n \subset R_n$ so $y$ satisfies an equation of integral dependence over $R_n$. Thus $T_n$ is a finite intersection of valuation rings of the field $K_n$. Hence $T_n$ is a Prüfer domain with quotient field $K_n$ and each valuation ring containing $T_n$ contains some $V_i \cap K_n$ [1, p. 132–134]. It follows that $T=\bigcup_{i=1}^{\infty} T_i$ is also Prüfer [2, p. 260], $T$ has quotient field $K$, and $T=\bigcap_{i=1}^{\infty} V_i$. Now suppose $W$ is a valuation ring between $T$ and $K$. Since $W$ contains $T_n$, $W$ contains some $V_i \cap K_n$. If $W$ contains $V_1 \cap K_n$ for all $n$, then $W$ contains $V_1$ so either $W = V_1$ or $W = K$. If $V_1 \cap K_n \not\subset W$, then for $s \geqq n$, let $W_s = W \cap K_s$. We know that $V_j \cap K_s \subset W_s$ for some $j < s$. But, for $j \geqq n$, $V_j \cap K_s \cap K_n = V_1 \cap K_n$, so $W_s$ is contained in $V_j \cap K_s$ for some $j < n$. Since $V_j$ has rank $j$, we see that $W_s$ has rank $< n$. It follows that $W = \bigcup_{s=n}^{\infty} W_s$ also has rank less than $n$.

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