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COGENERATOR ENDOMORPHISM RINGS

RONALD L. WAGONER[1]

ABSTRACT. If R is a ring and P is a finitely generated projective right R-module, what properties of R does the R-endomorphism ring of P inherit? Rosenberg and Zelinsky have shown that if R is quasi-Frobenius, and P also has every simple epimorphic image isomorphic to a submodule, then the R-endomorphism ring of P is also quasi-Frobenius. In this paper we show that if R is a cogenerator ring, and P is a finitely generated projective right R-module with every simple epimorphic image isomorphic to a submodule of P, then the R-endomorphism ring of P is also a cogenerator ring.

0. **Introduction.** If a right R-module P_R is a progenerator, and $S=\mathrm{End}_R(P)$, then R and S are categorically equivalent. However, if P_R is just finitely generated projective, surprisingly little is known about S.

In this connection, Rosenberg and Zelinsky [**5**] have shown that if R is quasi-Frobenius and P_R is a finitely generated projective right R-module with every simple epimorphic image isomorphic to a simple submodule, then $\mathrm{End}_R(P)$ is also quasi-Frobenius. We call a right R-module M_R an RZ module if every simple epimorphic image of M_R is isomorphic to a simple submodule of M_R.

In this paper we show

THEOREM. *If R is a cogenerator ring and P_R is a finitely generated projective RZ module, then* $\mathrm{End}_R(P)$ *is also a cogenerator ring.*

1. **Cogenerator endomorphism rings.** Throughout this paper R will denote an associative ring with identity, and J will denote its Jacobson radical.

We adopt the standard notation that M_R $({}_RM)$ means M is a right (left) R-module, and $N_R<M_R$ means N_R is a submodule of M_R. For $I_R<R_R$ and ${}_RI'<{}_RR$,

$$l_R(I_R) = \{x \in R \mid xI = 0\}, \qquad r_R({}_RI') = \{x \in R \mid I'x = 0\}.$$

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A ring R is a cogenerator ring if ${}_RR$ and R_R are cogenerators; equivalently, R is a cogenerator ring if ${}_RR$ and R_R are injective and for each ${}_RI < {}_RR$ and for each $I'_R < R_R$, $l_R r_R({}_RI) = {}_RI$ and $r_R l_R(I'_R) = I'_R$ [3].

Onodera [3] shows that if R is a cogenerator ring, then R is semiperfect. Hence

$${}_RR \simeq \bigoplus_{i=1}^{n} Re_i \quad \text{and} \quad R_R \simeq \bigoplus_{i=1}^{n} e_iR$$

where $\{e_1, \cdots, e_n\}$ is an orthogonal collection of primitive idempotents. Since a module ${}_RM$ is a cogenerator if, and only if, ${}_RM$ contains a copy of the injective envelope of each simple left R-module [4, Lemma 1], ${}_RR$ and R_R contain copies of the injective envelope of each simple left and right R-module respectively. Now let ${}_RU$ and ${}_RU'$ be simple and let $E({}_RU)$ and $E({}_RU')$ be their injective envelopes. Then ${}_RU \simeq {}_RU'$ if, and only if, $E({}_RU) \simeq E({}_RU')$. Hence, a simple counting argument shows that if R is a cogenerator ring and $\{f_1, \cdots, f_k\}$ is a basic set of primitive idempotents for R (for each primitive idempotent e of R, Re is isomorphic to exactly one of $Rf_1, \cdots, Rf_k$), then each Rf_i has a simple essential socle and there exists a permutation σ of $\{1, \cdots, k\}$ such that

$$\operatorname{soc}(Rf_i) \simeq Rf_{\sigma(i)}/Jf_{\sigma(i)}.$$

1.1. PROPOSITION. *Let R be a cogenerator ring and let e be a primitive idempotent in R. Then* $\operatorname{soc}(eR_R) \simeq fR/fJ$ *if, and only if,* $\operatorname{soc}({}_RRf) \simeq Re/Je$.

PROOF. Let $\operatorname{soc}(eR) \simeq fR/fJ$. Then f is also a primitive idempotent. Suppose $Re/Je \simeq \operatorname{soc}(Rg)$ and let $(\ )^*$ denote $\operatorname{Hom}_R(\ , R)$. Then

$$Re \longrightarrow Re/Je \longrightarrow 0$$

is exact, and ${}_RR$ is injective, so

$$0 \longrightarrow (Re/Je)^* \longrightarrow (Re)^*$$

is exact. Hence $\operatorname{soc}(eR) \simeq (Re/Je)^*$ (duals of simples are simple [2, Theorem 2]). So $(Re/Je)^* \simeq (\operatorname{soc}(Rg))^* \simeq \operatorname{soc}(eR)$.

Since $0 \to \operatorname{soc}(Rg) \to Rg$ is exact, $(Rg)^* \to (\operatorname{soc}(Rg))^* \to 0$ is also exact. Hence $gR/gJ \simeq (\operatorname{soc}(Rg))^* \simeq \operatorname{soc}(eR) \simeq fR/fJ$ and $gR \simeq fR$ so $Re/Je \simeq \operatorname{soc}(Rf)$.

By symmetry we get the converse.

If R is semiperfect and P_R is finitely generated projective, then $P_R \simeq \bigoplus_{j=1}^{m} e_jR$ with each e_j a primitive idempotent of R. In this case the basic submodule of P_R, denoted by $B(P)$, is

$$B(P) = \bigoplus_{i=1}^{t} f_i R$$

with each $f_i \in \{e_1, \cdots, e_m\}$ and for each $j \in \{1, \cdots, m\}$, $e_j R$ is isomorphic to exactly one of $f_1 R, \cdots, f_t R$. Since R is semiperfect, the basic submodule is unique up to isomorphism, and is isomorphic to a direct summand of R. We will write $B(P) = fR$ when $B(P) \simeq fR$ and f is an idempotent in R. If e is an idempotent of R and $B(eR) = eR$, we will say e is a basic idempotent.

1.2. COROLLARY. *Let R be a cogenerator ring and let e be a basic idempotent in R. Then* $\operatorname{soc}(eR_R) \simeq fR/fJ$ *if, and only if,* $\operatorname{soc}({}_R Rf) \simeq Re/Je$.

1.3. PROPOSITION. *Let R be a cogenerator ring and let P_R be finitely generated projective. Then the following are equivalent*:

(a) *P_R is an RZ module.*
(b) *$B(P) = eR$ and* $\operatorname{soc}(eR) \simeq eR/eJ$.
(c) *$B(P) = eR$ and* $\operatorname{soc}(Re) \simeq Re/Je$.
(*d*) $\operatorname{Hom}_R(P, R)$ *is an RZ module.*

PROOF. (a)$\Leftrightarrow$(*b*): P_R is an RZ module if, and only if, $B(P) = eR$ is an RZ module. A simple counting argument shows eR_R is an RZ module if, and only if, $\operatorname{soc}(eR) \simeq eR/eJ$.

(b)$\Leftrightarrow$(*c*): 1.2.

(c)$\Leftrightarrow$(d): Same as (a)$\Leftrightarrow$(b), since $B(P) = eR$ if, and only if, $B(\operatorname{Hom}_R(P, R)) = Re$.

1.4. THEOREM. *Let R be a cogenerator ring and let P_R be a finitely generated projective RZ module. Then* $\operatorname{End}_R(P)$ *is also a cogenerator ring.*

PROOF. Let

$$P \simeq \bigoplus_{i=1}^{n} e_i R \quad \text{and} \quad B(P) = eR = e_1 R \oplus \cdots \oplus e_k R$$

with each e_i a primitive idempotent.

By [1, Theorem 1.5] eRe and $\operatorname{End}_R(P)$ are categorically equivalent, hence we need only see that eRe is a cogenerator ring.

Now, $eRe \simeq \bigoplus_{i=1}^{k} eRe_i$ and each eRe_i is indecomposable since

$$eRe_i \simeq eR \underset{R}{\otimes} Re_i$$

and

$$\bigoplus_{i=1}^{k} Re_i \simeq Re \simeq Re \otimes eR \otimes Re \simeq \bigoplus_{i=1}^{k} Re \otimes eR \otimes Re_i.$$

Let $0 \neq eM < eRe_i$, then $0 \neq ReM < ReRe_i = Re_i$. Hence $\mathrm{soc}(Re_i) < ReM$ and so $e \cdot \mathrm{soc}(Re_i) < eReM = eM$. Since $\mathrm{soc}(Re) \simeq Re/Je$, $e \cdot \mathrm{soc}(Re_i) \neq 0$. Hence, for each $i = 1, \cdots, k$, $e \cdot \mathrm{soc}(Re_i)$ is a simple essential submodule of eRe_i. Let $E[e \cdot \mathrm{soc}(Re_i)]$ be the injective envelope of $e \cdot \mathrm{soc}(Re_i)$, then

$$eRe < \bigoplus_{i=1}^{k} E[e \cdot \mathrm{soc}(Re_i)] < \prod_{A} eR.$$

(If ${}_RR$ is a cogenerator then ${}_{eRe}eRe$ is also a cogenerator since

$$0 \to Re \otimes eM \to \prod R$$

exact, gives

$$0 \to eR \otimes Re \otimes eM \to eR \otimes \prod R$$

exact, and $eR \otimes Re \otimes eM \simeq eM$ and $eR \otimes \prod R \simeq \prod eR$.)

Let $e = (er_\alpha)_{\alpha \in A}$ and let L_R be the submodule of eR_R generated by $\{er_\alpha \mid \alpha \in A\}$. Then let $f \in \mathrm{Hom}_R(eR/L, R)$. Now, $eR/L = (e+L)eR$ and $L = eL + L = (e+L)eL$ so $0 = f(0) = f(e+L)eL$ hence $eR \cdot f(e+L)eL = 0$. But then $eR \cdot f(e+L)e \cdot e = 0$ in $\prod_A eR$, so $eR \cdot f(e+L)e = 0$. Since $\mathrm{soc}(Re) \simeq Re/Je$, $R \cdot f(e+L)e = 0$, so $f(e+L)e = 0 = f(e+L)$ and so $f = 0$. Now, $\mathrm{Hom}_R(eR/L, R) = 0$ and R_R is a cogenerator, so $eR = L$. Hence there exist elements $x_1, \cdots, x_m$ in R such that

$$\sum_{i=1}^{m} er_i x_i = e.$$

Let π_i be the projection of $\prod_A eR$ onto the ith coordinate then

$$\sum_{i=1}^{m} \pi_i x_i e \colon \prod_{A} eR \to eRe$$

via

$$(ey_\alpha)_{\alpha \in A} \to \sum_{i=1}^{m} ey_i x_i e$$

splits the embedding of eRe in $\prod_A eR$. Hence eRe is a direct summand of $\bigoplus_{i=1}^{k} E[e \cdot \mathrm{soc}(Re_i)]$ and so is injective and contains a copy of each simple left eRe-module. Hence, ${}_{eRe}eRe$ is an injective cogenerator.

Now, using Proposition 1.3, we can repeat the above arguments on the opposite side with Re, and get eRe_{eRe} is an injective cogenerator.

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Fresno State College, Fresno, California 93726