PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 28, Number 2, May 1971

ON LIE RINGS SATISFYING THE FOURTH ENGEL CONDITION

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ABSTRACT. In this paper we prove that a Lie ring of characteristic prime to 2, 3 and 5, satisfying the fourth Engel condition, is nilpotent.

1. Let L be a Lie ring satisfying the fourth Engel condition, that is, for any a, x in L, $ax^4=0$. Higgins [3] proved that if L has characteristic prime to 2, 3, 5, and 7, then L is nilpotent. Bachmuth et al. [1] showed that if L has characteristic 5, then L need not be nilpotent. The purpose of this note is to show that if L has characteristic prime to 2, 3 and 5, then L is nilpotent.

2. Let R be the (additive) endomorphism ring of L. Let D be the subset consisting of the inner derivations of L, $X: a\to ax$. D is a Lie ring under the product $(X, Y)=XY-YX$. D is easily seen to be a homomorphic image of L under the map $x\to X$.

THEOREM. *A Lie ring L of characteristic* 7, *satisfying the fourth Engel condition is nilpotent.*

PROOF. For any X, Y in D, the following relations are evident from Higgins [3, Theorem 4]:

$$X^3Y + 2XYX^2 = 0. \tag{1}$$

$$X^3Y - 4X^2YX + 6XYX^2 - 4YX^3 = 0. \tag{2}$$

$$X^3Y^3 = - Y^3X^3. \tag{3}$$

$$YX^3Y^2 = 0. \tag{4}$$

$$Y^2X^3Y = - X^3Y^3. \tag{5}$$

In the same paper Higgins discusses the process of linearization in his Lemma 1. Linearizing (1) we obtain,

$$\sum_{i=1}^{18} Y_{1\sigma_i}Y_{2\sigma_i}Y_{3\sigma_i}Y_{4\sigma_i} = 0. \tag{6}$$

The σ_i's are some fixed permutations of $\{1, 2, 3, 4\}$, and Y_1, Y_2, Y_3, Y_4 are any elements of D.

Received by the editors September 1, 1970.
AMS 1969 *subject classifications*. Primary 1730.
Key words and phrases. Lie rings, Engel condition, exponent 7.

We first show that $X^3Y^3=3(Y, X, X, X, Y, Y)$ and is hence in D.

$$(Y, X) = YX - XY,$$
$$(Y, X, X) = YX^2 - 2XYX + X^2Y.$$

Using (1) and (2) we obtain

$$\begin{aligned}(Y, X, X, X) &= YX^3 - 3XYX^2 + 3X^2YX - X^3Y \\ &= YX^3 + (-4X^2YX + 6XYX^2) - (2XYX^2 + X^3Y) \\ &= YX^3 + (-4X^2YX + 6XYX^2) \\ &= YX^3 + (4YX^3 - X^3Y) \\ &= 5YX^3 - X^3Y,\end{aligned}$$
$$(Y, X, X, X, Y) = 6YX^3Y - X^3Y^2 - 5Y^2X^3.$$

Using (3), (4) and (5) we obtain,

$$\begin{aligned}&(Y, X, X, X, Y, Y) \\ &\quad = 6YX^3Y^2 - X^3Y^3 - 5Y^2X^3Y - 6Y^2X^3Y + YX^3Y^2 + 5Y^3X^3 \\ &\quad = 5X^3Y^3.,\end{aligned}$$

Hence, $X^3Y^3=15X^3Y^3=3(Y, X, X, X, Y, Y)$ is in D.

Given any eight elements $X_1, \cdots, X_8$ of D, let

$$A_1 = X_1^3X_2^3, \quad A_2 = X_3^3X_4^3, \quad A_3 = X_5^3X_6^3, \quad A_4 = X_7^3X_8^3.$$

Then each A is in D. Moreover, using (3) we have,

$$A_1A_2 = X_1^3X_2^3X_3^3X_4^3 = X_3^3X_1^3X_2^3X_4^3 = X_3^3X_4^3X_1^3X_2^3 = A_2A_1.$$

Substituting A_j for Y_j in (6) and using the commutativity of the A_j's we have

$$18A_1A_2A_3A_4 = 0.$$

Dividing by 18 we have

$$A_1A_2A_3A_4 = 0.$$

Consequently,

$$X_1^3X_2^3 \cdots X_8^3 = 0.$$

It follows from Lemma 4 in Higgins [3] and his remark near the end, that L is nilpotent.

In his thesis [4, p. 92] Walkup proved the following lemma, here stated in a special context:

LEMMA. *Suppose:*

(1) *every Lie ring of characteristic prime to certain primes* $p_1, p_2, \cdots, p_s$ *satisfying the identical relations* f_i *is nilpotent;*

(2) *every Lie ring of characteristic* p_s *satisfying the same identical relations* f_i *is nilpotent.*

Then every Lie ring of characteristic prime to $p_1, p_2, \cdots, p_{s-1}$ *satisfying the identical relations* f_i *is nilpotent.*

As a consequence we obtain the following corollary:

COROLLARY 1. *A Lie ring of characteristic prime to* 2, 3 *and* 5 *satisfying the fourth Engel condition is nilpotent.*

Let G be a group of exponent 7 satisfying the fourth Engel congruence, $(g, h, h, h, h) \equiv 1 \bmod G_6$. The associated Lie ring of G (see Hall [2] for definition) is of characteristic 7 and satisfies the fourth Engel condition and is therefore nilpotent. We have shown the following:

COROLLARY 2. *The associated Lie ring of a group of exponent* 7, *satisfying the fourth Engel congruence, is nilpotent.*

The author is indebted to Dr. H. Y. Mochizuki and Dr. S. Bachmuth for many stimulating discussions.

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