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A NOTE ON TWO-SIDED IDEALS IN C^*-ALGEBRAS[1]

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ABSTRACT. An elementary proof is given of the fact that $(I+J)^+ = I^+ + J^+$ for I and J closed two-sided ideals in a C^*-algebra.

In this note we give a short and elementary proof of the following theorem, which was proposed as a problem by J. Dixmier [**1**, Problem 1.9.12] and first proved by E. Størmer [**4**].

THEOREM. *Let A be a C^*-algebra. If I and J are uniformly closed two-sided ideals in A then* $(I+J)^+ = I^+ + J^+$.

Here I^+ denotes the set of positive elements in I. To prove the theorem we may assume that A has an identity, denoted by e. In [**4**] Størmer proved the theorem by using some results of E. G. Effros [**2**] and some quite delicate calculations using the functional calculus to prove the following lemma, from which the theorem follows by a straightforward induction argument.

LEMMA. *With the assumptions as in the theorem, let a belong to $(I+J)^+$, and let $\epsilon > 0$ be given. Then there exists b in I^+ and c in J^+ such that* $0 \leqq a - b - c \leqq \epsilon e$.

A SHORT PROOF OF THE LEMMA. Let a belong to $(I+J)^+$. Then $a = f + g$ for some f in I and g in J. Since $a = a^*$, we may assume that $f = f^*$ and $g = g^*$. Let $h = |f| + |g| + \epsilon e$. Then h is invertible and $0 \leqq a \leqq h$. Let $d = a^{1/2}h^{-1/2}$. Then d is in A, and $0 \leqq d^*d = h^{-1/2}ah^{-1/2} \leqq h^{-1/2}hh^{-1/2} = e$. Thus $dd^* \leqq e$, and $0 \leqq a - d|f|d^* - d|g|d^* = \epsilon dd^* \leqq \epsilon e$. Since $d|f|d^*$ is in I^+ and $d|g|d^*$ is in J^+ the lemma is proved.

We note that G. K. Pedersen has given another proof of this theorem [**3**].

REFERENCES

1. J. Dixmier, *Les C^*-algèbres et leurs représentations*, Cahiers Scientifiques, fasc. 29, Gauthier-Villars, Paris, 1964. MR **30** #1404.

2. E. G. Effros, *Order ideals in a C^*-algebra and its dual*, Duke Math. J. **30** (1963), 391–412.

3. G. K. Pedersen, *A decomposition theorem for C^*-algebras*, Math. Scand. **22** (1968), 266–268.

4. E. Størmer, *Two-sided ideals in C^*-algebras*, Bull. Amer. Math. Soc. **73** (1967), 254–257. MR **34** #8210.

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