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COMMUTING OPERATOR SOLUTIONS OF ALGEBRAIC EQUATIONS

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ABSTRACT. Let $G(w, z)$ be a complex polynomial, and S a bounded operator of scalar type on a complex Banach space, whose spectrum avoids the points λ for which $G(\lambda, z)=0$ has multiple roots z. The form of a bounded operator T which commutes with S and satisfies $G(S, T)=0$ is established.

1. **Introduction.** Fix a Banach space X over the complex numbers $\mathbf{C}$, and let $\mathcal{B}$ denote the Banach algebra of all bounded linear operators on X. Given $S \in \mathcal{B}$ of scalar type, and given a polynomial in two indeterminates

$$G(w, z) = a_n(w)z^n + \cdots + a_1(w)z + a_0(w) \qquad (a_i(w) \in \mathbf{C}[w]),$$

we seek operators $T \in \mathcal{B}$ such that

$$\text{(E)} \qquad T \text{ commutes with } S \text{ and } G(S, T) = 0.$$

Denoting the spectrum of S by σ, we assume:

$$\text{(A)} \qquad \begin{array}{l} \text{For each } \lambda \in \sigma, \text{ the polynomial } G(\lambda, z) \text{ has} \\ n \text{ distinct complex roots } t_1(\lambda), \cdots, t_n(\lambda). \end{array}$$

We shall establish:

THEOREM. *Notation and assumptions as above, there exist* $T_1, \cdots, T_n \in \mathcal{B}$ *satisfying* (E), *and having the following property*: $T \in \mathcal{B}$ *satisfies* (E) *if and only if there exist* $F_1, \cdots, F_n \in \mathcal{B}$ *such that*

$$\text{(D)} \qquad \begin{array}{l} F_i \text{ commutes with } S, \\ F_i^2 = F_i, \quad F_iF_j = 0 \text{ for } i \neq j, \quad \Sigma_i F_i = I, \\ T = \Sigma_i T_i F_i. \end{array}$$

An operator S is of *scalar type* [1, p. 332] if S admits a resolution of the identity $E(\cdot)$, and if moreover S can be recovered from $E(\cdot)$ by integration over σ:

$$S = \int \lambda E(d\lambda).$$

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In particular, a normal operator on a Hilbert space is of scalar type. In the theorem, G need not be irreducible, but by (A) no repeated factors are permitted in the prime decomposition of G. The idempotents F_i are not asserted to be values of $E(\cdot)$, and indeed need not be, if S has multiplicity greater than 1.

Foguel [2] proved this theorem for the special case $G(w, z) = g(z) - w$, g a complex polynomial, and we imitate his proof. For a given solution T of (E), the main step in constructing the F_i is to check that $G_2(S, T)$ is invertible, where $G_2(w, z) = \partial G(w, z)/\partial z$. In Foguel's case, $G_2(w, z) = g'(z)$ is independent of w, and the existence of $g'(T)^{-1}$ follows immediately from (A) and the spectral mapping theorem. The general proof below uses maximal ideals, and we are indebted to the referee for a substantial simplification of our original argument.

2. **Proof of the theorem.** Let $\mathfrak{M}$ denote the Banach algebra of all essentially bounded measurable complex functions on σ.

LEMMA 1. *There exist* $t_1, \cdots, t_n \in \mathfrak{M}$ *such that, for each* $\lambda \in \sigma$,

(1) $G(\lambda, z) = a_n(\lambda)\Pi_j(z - t_j(\lambda))$,

(2) $G_2(\lambda, z) = a_n(\lambda)\Sigma_i\Pi_{j\neq i}(z - t_j(\lambda))$.

PROOF. Let K be an oriented cut in $\mathbf{C}$ joining the singularities $\lambda_1, \cdots, \lambda_k$ of the algebraic function(s) determined by G. Then the roots $t_i(\lambda)$ can be chosen to be holomorphic in $\mathbf{C} - K$. If each t_i is analytically continued to $K' = K - \{\lambda_1, \cdots, \lambda_k\}$ from the left, say, then the extended t_i are defined and locally bounded on $\mathbf{C}' = \mathbf{C} - \{\lambda_1, \cdots, \lambda_k\}$, have, at worst, jump discontinuities from the right along K', and have for values $\{t_i(\lambda)\}$ precisely the set of roots of $G(\lambda, z)$, for each $\lambda \in \mathbf{C}'$. Assumption (A) provides that $\sigma \subset \mathbf{C}'$, and in particular that $a_n(\lambda) \neq 0$ for $\lambda \in \sigma$; hence restricting the t_i to σ establishes (1), from which (2) is immediate.

By [1, Lemma 6, p. 341], the map $\mathfrak{M} \to \mathfrak{B}$ given by $f \mapsto f(S) \equiv \int f(\lambda)E(d\lambda)$ is a continuous algebra homomorphism. Clearly it extends by $z \mapsto z$ to a homomorphism $\mathfrak{M}[z] \to \mathfrak{B}[z]$, which carries (1) and (2) over to relations

(3) $G(S, z) = a_n(S)\Pi_j(z - T_j)$,

(4) $G_2(S, z) = a_n(S)\Sigma_i\Pi_{j\neq i}(z - T_j)$,

in which we have set $T_j = t_j(S)$. Then each T_j commutes with S and, by (3), obeys $G(S, T_j) = 0$.

Now suppose that T satisfies (E). Then in particular T must commute with $E(\cdot)$ [1, Theorem 5, p. 329] and hence with the T_j, so that $z \mapsto T$ defines a homomorphism $\mathfrak{B}[z] \to \mathfrak{B}$, which carries (3) and (4) to

(5) $0 = G(S, T) = a_n(S)\Pi_j(T - T_j)$,
(6) $G_2(S, T) = a_n(S)\Sigma_i\Pi_{j\neq i}(T - T_j)$.

LEMMA 2. *$G_2(S, T)^{-1}$ exists in $\mathfrak{B}$ and commutes with S and T.*

PROOF. Let $\mu: \mathfrak{A} \to \mathbf{C}$ denote any nonzero continuous homomorphism, where $\mathfrak{A} \subset \mathfrak{B}$ is the (commutative) full algebra generated by $E(\cdot)$ and T [**1**, p. 342]. Then $\lambda = \mu(S) \in \sigma$, for otherwise $(S - \lambda I)^{-1} \in \mathfrak{A}$ by definition of "full algebra", and $I = (S - \lambda I)(S - \lambda I)^{-1}$ would go by μ to $1 = 0$. Hence $G(\lambda, \mu(T)) = \mu(G(S, T)) = 0$, and then $\mu(G_2(S, T)) = G_2(\lambda, \mu(T)) \neq 0$, by (A). Thus $G_2(S, T)$ lies in no maximal ideal of $\mathfrak{A}$, so is invertible in $\mathfrak{A}$.

Now for each i, set

$$F_i = G_2(S, T)^{-1} a_n(S) \Pi_{j\neq i}(T - T_j). \tag{7}$$

To verify (D), notice that $\Sigma_i F_i = I$ follows from (6). For $i \neq j$, $F_i F_j$ contains each factor $T - T_k$ at least once, so vanishes by (5); and then $F_i^2 = F_i(\Sigma_j F_j) = F_i$ follows. Similarly (7) and (5) give $(T - T_i)F_i = 0$ for each i, and summing yields $T = \Sigma_i T_i F_i$, to conclude the "only if" part of the proof.

Conversely, suppose that $F_1, \cdots, F_n$ obey (D). Then each F_i commutes with $E(\cdot)$, hence with each T_j, and $T = \Sigma_i T_i F_i$ commutes with S. Moreover, it follows by induction that $T^k F_i = T_i^k F_i$, hence that $G(S, T)F_i = 0$, for each i. Summing yields $G(S, T) = 0$, to conclude the "if" part.

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