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ON THE EXISTENCE OF DOUBLE SINGULAR INTEGRALS FOR KERNELS WITHOUT SMOOTHNESS

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ABSTRACT. Calderón and Zygmund have proved the pointwise convergence of singular integrals in R^n for locally integrable homogeneous kernels whose even part is locally in $L \log L$ by change to polar coordinates and use of the boundedness in L^p of the maximal operator of the one-dimensional Hilbert transformation. The present note shows how analogous results for double singular integrals can be derived from boundedness of the maximal operator of the double Hilbert transform.

For $i=1, 2$ let K_i be a complex valued function defined in R^{n_i} which is (positively) homogeneous of degree $-n_i$, i.e., $K_i(\lambda x_i) = \lambda^{-n_i} K_i(x_i)$ for $x_i \neq 0$, $\lambda > 0$, locally integrable away from the origin, of mean value zero on the unit sphere of R^{n_i}, i.e.,

$$\int_{|x_i'|=1} K_i(x_i')dx_i' = 0$$

(where dx_i' denotes ordinary surface measure on $S^{n_i-1} = \{x_i' : |x_i'| = 1\}$) and whose even part belongs to $L \log L$ on the unit sphere, i.e.,

$$(1) \quad \int_{|x_i'|=1} | K_i(x_i') + K(-x_i') | \log^+ | K_i(x_i') + K_i(-x_i') | \, dx_i' < \infty .$$

A. Zygmund called attention to the problem of showing by the methods of [2] that if

$$(2) \qquad f^*(x) = \sup\{ | \tilde{f}_{\epsilon_1,\epsilon_2}(x) | : \epsilon_1, \epsilon_2 > 0\}$$

where

$$\tilde{f}_{\epsilon_1,\epsilon_2}(x_1, x_2) = \int_{|x_1-y_1|>\epsilon_1} \int_{|x_2-y_2|>\epsilon_2} K_1(x_1 - y_1)K_2(x_2 - y_2)f(y_1, y_2)dy_2dy_1$$

then

$$(3) \qquad \|f^*\|_p \leqq A_p\|f\|_p \qquad \text{for } 1 < p < \infty$$

where A_p depends on p, K_1, K_2. In case the moduli of continuity ω_i of

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K_1, K_2 restricted to S^{n_1-1}, S^{n_2-1} satisfy the Dini condition $\int_0^1 t^{-1}\omega_i(t)dt < \infty$ for $i=1, 2$, this was shown by Cotlar in [3]. The purpose of this note is to prove the following

PROPOSITION. *Suppose K_1, K_2 are homogeneous of degree $-n_1$, $-n_2$, respectively, locally integrable and of mean value zero on S^{n_i-1} and satisfy* (1), *then for f^* defined by* (2), (3) *is valid. Moreover if ν indicates how many of K_1, K_2 are odd ($\nu=0, 1, 2$) then $A_p=O((p-1)^{\nu-4})$ as $p\downarrow 1$ ($O(p^{4-\nu})$ as $p\to\infty$).*

The proof requires the following

LEMMA. *Let $f\in L^p(R^2)$ and*

$$\bar{f}(\xi_1, \xi_2) = \sup_{\eta_1,\eta_2>0}\left|\pi^{-2}\int_{|\xi_1-\tau_1|>\eta_1}\int_{|\xi_2-\tau_2|>\eta_2}(\xi_1-\tau_1)^{-1}(\xi_2-\tau_2)^{-1}\bar{f}(\tau_1, \tau_2)d\tau_2 d\tau_1\right|$$

then $\|\bar{f}\|_p \leqq A_p\|f\|_p$ where $A_p=O((p-1)^{-2})$ for $p\downarrow 1$.

For the maximal double conjugate function of a periodic function the analogous assertion follows from the arguments of [6, especially pp. 228–233] and with $A_p=((p-1)^{-4})$ is Theorem 3 of [4]. Again with $A_p=O((p-1)^{-4})$ the lemma is contained in [3, Theorem 3, p. 102]. A proof analogous to that of Theorem 6′ of [6] might run briefly as follows.

Let

$$P(\xi, \eta) = \pi^{-1}\eta(\xi^2+\eta^2)^{-1}, \qquad Q(\xi, \eta) = \pi^{-1}\xi(\xi^2+\eta^2)^{-1}$$

then $(i\pi)^{-1}(\xi-i\eta)^{-1}=P(\xi,\eta)-iQ(\xi,\eta)$, hence

$$[P(\cdot, \eta_1)\otimes P(\cdot, \eta_2) - Q(\cdot, \eta_1)\otimes Q(\cdot, \eta_2)] * f(\xi_1, \xi_2)$$

and

$$-[P(\cdot, \eta_1)\otimes Q(\cdot, \eta_2) + Q(\cdot, \eta_1)\otimes P(\cdot, \eta_2)] * f(\xi_1, \xi_2)$$

are the real and imaginary parts, respectively, of

$$F(\zeta_1, \zeta_2) = (i\pi)^{-2}\int\int f(\tau_1, \tau_2)(\zeta_1-\tau_1)^{-1}(\zeta_2-\tau_2)^{-1}d\tau_1 d\tau_2 \qquad (\zeta_j = \xi_j + i\eta_j).$$

It will be seen that F is in H^p. In what follows C will denote a constant not necessarily the same at each occurrence. It is well known that, e.g.,

$$\|Q(\cdot,\eta_1)*f(\cdot,\xi_2)\|_p \leqq Cpp'\|f(\cdot,\xi_2)\|_p \qquad ((p')^{-1}+p^{-1}=1)$$

hence $\|F\|[H^p] \leqq C(pp')^2\|f\|_p$, and so, if

$$F^*(\xi_1,\xi_2) = \sup\{\,|F(\xi_1+i\eta_1,\xi_2+i\eta_2)|:\eta_1,\eta_2>0\}$$

then $\|F^*\|_p \leqq C(pp')^2\|f\|_p$. Also it is well known that

$$\|\sup\{P(\cdot,\eta_1)\otimes P(\cdot,\eta_2)*|f|:\eta_1,\eta_2>0\}\|_p \leqq C(pp')^2\|f\|_p.$$

Hence consideration of the real part of F leads to

$$\|\sup\{\,|Q(\cdot,\eta_1)\otimes Q(\cdot,\eta_2)*f|:\eta_1,\eta_2>0\}\|_p \leqq C(pp')^2\|f\|_p.$$

It remains to observe that if $H(\xi,\eta)=(\pi\xi)^{-1}(1-\chi_{(-\eta,\eta)})$, $\chi_{(-\eta,\eta)}$ being the characteristic function of the interval $(-\eta,\eta)$, then

$$\begin{aligned}&H(\cdot,\eta_1)\otimes H(\cdot,\eta_2)-Q(\cdot,\eta_1)\otimes Q(\cdot,\eta_2)\\&\quad=(H(\cdot,\eta_1)-Q(\cdot,\eta_1))\otimes H(\cdot,\eta_2)+Q(\cdot,\eta_1)\otimes(H(\cdot,\eta_2)-Q(\cdot,\eta_2))\end{aligned}$$

and $|H(\xi,\eta)-Q(\xi,\eta)| \leqq \eta^{-1}\psi(\xi\eta^{-1})$ where ψ is even, nonincreasing in $(0,\infty)$ and integrable so that, e.g.,

$$\left\|\sup_{\eta_1>0}\eta_1^{-1}\psi(\eta_1^{-1}\,\cdot)*g(\cdot,\xi_2)\right\|_p \leqq Cpp'\|g(\cdot,\xi_2)\|_p$$

(e.g., by Lemma 1 of Chapter II of [1]) where

$$g(\xi_1,\xi_2)=\sup_{\eta_2>0}|H(\cdot,\eta_2)*f(\xi_1,\cdot)(\xi_2)|.$$

Proof of the Proposition. First of all, for a.e. (x_1,x_2) and any $\epsilon_1,\epsilon_2>0$,

$$(4)\qquad \int_{|y_1|>\epsilon_1}\int_{|y_2|>\epsilon_2}|K_1(y_1)K_2(y_2)f(x_1-y_1,x_2-y_2)|\,dy_2dy_1<\infty.$$

This follows as in [2, p. 292] by integration of the last integral over any compact subset of $R^{n_1}\times R^{n_2}$. In fact, let $B_i=\{x_i:x_i\in R^{n_i},\ |x_i|\leqq r_i\}$ then the integral of the left-hand side of (4) over $B_1\times B_2$ is at most

$$\begin{aligned}\int_{|y_1'|=1}\int_{|y_2'|=1}&|K_1(y_1')K_2(y_2')|\int_{B_1}\int_{B_2}\int_{\epsilon_1}^{\infty}\int_{\epsilon_2}^{\infty}\\&\cdot|f(x_1-y_1't_1,x_2-y_2't_2)|\,t_1^{-1}t_2^{-1}dt_2dt_1dx_2dx_1dy_2'\,dy_1'\\&\leqq C\epsilon_1^{-1/p}\epsilon_2^{-1/p}r_1^{n_1/p'+1/p}r_2^{n_2/p'+1/p}\|f\|_p.\end{aligned}$$

If K_1, K_2 are both odd then

$$\begin{aligned}\tilde{f}_{\epsilon_1,\epsilon_2}(x_1, x_2) &= -\int_{|y_1|>\epsilon_1}\int_{|y_2|>\epsilon_2} K_1(y_1)K_2(y_2)f(x_1 - y_1, x_2 + y_2)dy_2dy_1 \\ &= -\int_{|y_1|>\epsilon_1}\int_{|y_2|>\epsilon_2} K_1(y_1)K_2(y_2)f(x_1 + y_1, x_2 - y_2)dy_2dy_1 \\ &= \int_{|y_1|>\epsilon_1}\int_{|y_2|>\epsilon_2} K_1(y_1)K_2(y_2)f(x_1 + y_1, x_2 + y_2)dy_2dy_1.\end{aligned}$$

Hence by (4)

$$\tag{5}\begin{aligned}&\tilde{f}_{\epsilon_1,\epsilon_2}(x_1, x_2) \\ &\quad = (1/4)\int_{|y_1'|=1}\int_{|y_2'|=1} K_1(y_1')K_2(y_2')\tilde{f}_{\epsilon_1,\epsilon_2}(x_1, x_2; y_1', y_2')dy_2'\,dy_1'\end{aligned}$$

where

$$\tilde{f}_{\epsilon_1,\epsilon_2}(x_1, x_2; y_1', y_2') = \int_{|t_1|>\epsilon_1}\int_{|t_2|>\epsilon_2} f(x_1 - y_1't_1, x_2 - y_2't_2)t_1^{-1}t_2^{-1}dt_2dt_1.$$

Let

$$\tilde{f}(x_1, x_2; y_1', y_2') = \sup_{\epsilon_1,\epsilon_2>0} \left| \tilde{f}_{\epsilon_1,\epsilon_2}(x_1, x_2; y_1', y_2') \right| .$$

$\tilde{f}(\cdot, \cdot; y_1', y_2')$ restricted to any plane parallel to y_1' and y_2' is the maximal function of the truncated ordinary double Hilbert transforms of f restricted to such planes. Consequently by the lemma

$$\begin{aligned}&\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \tilde{f}(x_1 - y_1't_1, x_2 - y_2't_2; y_1', y_2')^p dt_2dt_1 \\ &\qquad\qquad \leqq A_p^p \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left| f(x_1 - y_1't_1, x_2 - y_2't_2) \right|^p dt_2dt_1\end{aligned}$$

and integration of this inequality over the space of planes parallel to y_1' and y_2' gives $\|\tilde{f}(\cdot\,; y_1', y_2')\|_p \leqq A_p\|f\|_p$. (2) now follows from (5) and Minkowski's inequality for integrals as in [2].

If K_1, K_2 are not both odd functions it appears sufficient to consider the case when both are even; if one is odd and the other even the following argument simplifies in an obvious manner. If as in [2, p. 299] ϕ denotes a continuously differentiable function of the real variable t, $t \geqq 0$, equal to zero in $(0, \frac{1}{4})$ and to 1 in $(\frac{3}{4}, \infty)$ then

$$\tag{6} \begin{aligned} \tilde{f}_{\epsilon_1,\epsilon_2}(x_1, x_2) = \Big(\int_{R^{n_1}} \int_{R^{n_2}} - \int_{R^{n_1}} \int_{|x_2-y_2|<\epsilon_2} \\ - \int_{|x_1-y_1|<\epsilon_1} \int_{R^{n_2}} + \int_{|x_1-y_1|<\epsilon_1} \int_{|x_2-y_2|<\epsilon_2} \Big) \\ \cdot K_1(x_1 - y_1)\phi(|x_1 - y_1| \epsilon_1^{-1}) \\ \cdot K_2(x_2 - y_2)\phi(|x_2 - y_2| \epsilon_2^{-1}) f(y_1, y_2) dy_2 dy_1. \end{aligned}$$

The integrand is integrable in $R^{n_1+n_2}$ for a.e. (x_1, x_2) by (4). Let R denote the (vector valued) Riesz kernel in R^{n_1} or R^{n_2} according to the context and define $(n_1 \times 1)$, $(1 \times n_2)$ and $(n_1 \times n_2)$ vector valued functions

$$\begin{aligned} g_{10}(x_1, x_2) &= - \text{ p.v. } R * f(\cdot, x_2)(x_1), \\ g_{01}(x_1, x_2) &= - \text{ p.v. } R * f(x_1, \cdot)(x_2), \\ g_{11}(x_1, x_2) &= \text{p.v. } (R \otimes R) * f(x_1, x_2). \end{aligned}$$

According to the lemma on pp. 299–300 of [2] if $K_{i1} = \text{p.v. } R * K_i$, $K_{i2} = \text{p.v. } R * (K_i \phi(|\cdot|))$ then K_{i1}, K_{i2} are odd, K_{i1} is homogeneous of degree $-n_i$, for $|x_i| \geqq 1$

$$| K_{i1}(x_i) - K_{i2}(x_i) | \leqq C \int_{|y_i'|=1} | K_i(y_i') | dy_i' \, | x_i |^{-n_i-1}$$

and there are functions G_i homogeneous of degree 0 such that for $|x_i| \leqq 1$, $|K_{i2}| \leqq G_i$ and $\int_{|x_i'|=1} G_i(x_i') dx_i' < \infty$. Then by (5.10) of [2] the first integral in (6) equals

$$\tag{7} \epsilon_2^{-n_2} \int K_1(x_1 - y_1)\phi(|x_1 - y_1| \epsilon_1^{-1}) \cdot \int K_{22}((x_2 - y_2)\epsilon_2^{-1}) g_{01}(y_1, y_2) dy_2 dy_1$$

and by §5 of [2] as a function of (y_1, x_2) the inner integral is in L^p, hence for a.e. $x_2^0 \in R^{n_2}$ the restriction to $x_2 = x_2^0$ is in $L^p(R^{n_1})$ and hence again by (5.10) of [2] (7) equals

$$\epsilon_1^{-n_1}\epsilon_2^{-n_2} \int\int [K_{12}((x_1 - y_1)\epsilon_1^{-1}) \otimes K_{22}((x_2 - y_2)\epsilon_2^{-1})] \cdot g_{11}(y_1, y_2) dy_1 dy_2.$$

A similar procedure with the second and third terms on the right-hand side of (6) leads to

$$
\begin{aligned}
&\left| \tilde{f}_{\epsilon_1,\epsilon_2}(x_1, x_2) \right| \epsilon_1^{n_1} \epsilon_2^{n_2} \\
&\quad \leqq \left| \iint \left[K_{12}((x_1 - y_1)\epsilon_1^{-1}) \otimes K_{22}((x_2 - y_2)\epsilon_2^{-1})\right] \cdot g_{11}(y_1, y_2) dy_1 dy_2 \right| \\
&\qquad + C \int_{|x_2 - y_2| < \epsilon_2} \left| K_2\left(\frac{x_2 - y_2}{|x_2 - y_2|}\right) \right| \\
&\qquad\qquad \cdot \left| \int K_{12}((x_1 - y_1)\epsilon_1^{-1}) g_{10}(y_1, y_2) dy_1 \right| dy_2 \\
&\qquad + C \int_{|x_1 - y_1| < \epsilon_1} \left| K_1\left(\frac{x_1 - y_1}{|x_1 - y_1|}\right) \right| \\
&\qquad\qquad \cdot \left| \int K_{22}((x_2 - y_2)\epsilon_2^{-1}) g_{01}(y_1, y_2) dy_2 \right| dy_1 \\
&\qquad + C \int_{|x_1 - y_1| < \epsilon_1} \int_{|x_2 - y_2| < \epsilon_2} \left| K_1\left(\frac{x_1 - y_1}{|x_1 - y_1|}\right) K_2\left(\frac{x_2 - y_2}{|x_2 - y_2|}\right) \right| \\
&\qquad\qquad \cdot |f(y_1, y_2)| \, dy_2 dy_1.
\end{aligned}
$$

The first term on the right-hand side is at most

$$
\begin{aligned}
&\epsilon_1^{n_1} \epsilon_2^{n_2} \left| \int_{|x_1 - y_1| > \epsilon_1} \int_{|x_2 - y_2| > \epsilon_2} (K_{11}(x_1 - y_1) \otimes K_{21}(x_2 - y_2)) \right. \\
&\qquad\qquad \left. \cdot g_{11}(y_1, y_2) dy_2 dy_1 \right| \\
&+ \epsilon_2^{n_2} \left(\int_{|x_1 - y_1| < \epsilon_1} G_1\left(\frac{x_1 - y_1}{|x_1 - y_1|}\right) + C \int_{|x_1 - y_1| > \epsilon_1} \left| (x_1 - y_1)\epsilon_1^{-1} \right|^{-n_1 - 1} \right) \\
&\qquad\qquad \cdot \left| \int_{|x_2 - y_2| > \epsilon_2} K_{21}(x_2 - y_2) \cdot g_{11}(y_1, y_2) dy_2 \right| dy_1 \\
&+ \int_{|x_1 - y_1| < \epsilon_1} \int_{|x_2 - y_2| < \epsilon_2} G_1\left(\frac{x_1 - y_1}{|x_1 - y_1|}\right) \left| (x_2 - y_2)\epsilon_2^{-1} \right|^{-n_2 - 1} \\
&\qquad\qquad \cdot |g_{11}(y_1, y_2)| \, dy_2 dy_1
\end{aligned}
$$

+ (3 similar terms obtained by interchanging x_1, y_1, ϵ_1, G_1, n_1 with x_2, y_2, ϵ_2, G_2, n_2 in the preceding 3 integrals)

$$+\int_{|x_1-y_1|<\epsilon_1}\int_{|x_2-y_2|<\epsilon_2} G_1\left(\frac{x_1-y_1}{|x_1-y_1|}\right)G_2\left(\frac{x_2-y_2}{|x_2-y_2|}\right) \cdot |g_{11}(y_1,y_2)|\,dy_2dy_1$$

$$+C\int_{|x_1-y_1|>\epsilon_1}\int_{|x_2-y_2|>\epsilon_2} |(x_1-y_1)\epsilon_1^{-1}|^{-n_1-1}|(x_2-y_2)\epsilon_2^{-1}|^{-n_2-1} \cdot |g_{11}(y_1,y_2)|\,dy_2dy_1.$$

Substitution in the estimate for $\tilde{f}_{\epsilon_1,\epsilon_2}(x_1,x_2)$, the result for products of odd kernels, Theorems 1 and 6 of [2],

$$\|g_{10}\|_p, \|g_{01}\|_p \leqq Cpp'\|f\|_p, \quad \|g_{11}\|_p \leqq C(pp')^2\|f\|_p$$

and the fact that the "outer" operators (in all but the first term) are positive imply (3).

REMARK. Completely analogously it can be shown by induction that if $K_i \in L_{\mathrm{loc}}(R^{n_i}-\{0\})$, $1 \leqq i \leqq N$, are several kernels all satisfying the conditions of the proposition then

$$f^*(x) = \sup_{\epsilon_i>0}\left|\int_{|x_1-y_1|>\epsilon_1}\cdots\int_{|x_N-y_N|>\epsilon_N}(K_1\otimes\cdots\otimes K_N)(x-y) \cdot f(y)dy_1,\cdots,dy_N\right|$$

satisfies (3), where $x=(x_1,\cdots,x_N)\in R^{n_1+\cdots+n_N}$.

It is also clear that analogous results hold for products of several kernels of any of the types discussed in [2].

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