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SEMIGROUPS ON ACYCLIC PLANE CONTINUA

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ABSTRACT. It is shown that an acyclic irreducible plane continuum which admits the structure of a topological semigroup is an arc if it has an identity, and is either an arc, is trivial, or is decomposible into an arc if it satisfies $M^2=M$. This extends some results of Friedberg and Mahavier concerning semigroups on chainable continua.

Let M be a topological semigroup with minimal ideal K whose underlying space is a nondegenerate compact metric continuum. If M has an identity, M is called a clan.

Under the assumption that M is chainable, Friedberg and Mahavier [3] showed that if M is a clan it is an arc, and if $M^2=M$ then either M is trivial, M is an arc, or $M|K$ is an arc and M is irreducible from a one-sided identity to some point. In this note we extend these results (using essentially the same arguments) by replacing the condition that M be chainable by the condition that M be an acyclic (i.e., contains no simple closed curve) plane continuum which is irreducible between two points. (Every nondegenerate chainable continuum is homeomorphic to such a continuum.)

THEOREM 1. *If M is an acyclic clan in the plane, then M is arcwise connected.*

PROOF. Let G be a closed subgroup of M with identity e and let $C(e)$ be the component of G containing e. $C(e)$ is a subcontinuum of M and is a group. Suppose $C(e)$ is nondegenerate. Then it is homogeneous and by [4] contains an arc; so by [1] it is a simple closed curve, contradicting the assumption that M is acyclic. Thus $C(e)$ is degenerate and G is totally disconnected. Then M is arcwise connected by [6].

COROLLARY. *If M is an acyclic plane continuum which is irreducible between two of its points it is an arc.*

REMARK. The referee has observed that except for the existence of the one-sided identity, the conclusion of the next theorem follows from Hunter's argument in [5, Theorem 8], without the assumption

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that M be acyclic. Also, a simplification suggested by the referee has been employed in the next argument.

THEOREM 2. *If M is an acyclic plane continuum which is irreducible between two of its points and $M^2=M$, then either*

(1) *$M=K$ and the multiplication on K is trivial,*

(2) *M is an arc, or*

(3) *M has a one-sided identity e, $M|K$ is an arc, and M is irreducible from e to some point.*

PROOF. Let E denote the set of idempotent elements of M, and for e in E, let H_e be the maximal subgroup containing e. Since M is acyclic, K is not the cartesian product of two nondegenerate continua [**5**, Lemma 2, p. 238]; so K is a group or multiplication in K is trivial [**7**, Corollary 1]. As in the proof of Theorem 1, if K is a group it is degenerate. In either case multiplication in K is trivial and K is a subset of E.

Now assume that $M\neq K$ and M is not an arc. Suppose M has no one-sided identity. Since M is irreducible between two points a and b, there exist points e and f in $E\backslash K$ such that $a\in H_e$, $b\in H_f$, H_e and H_f are connected, and $M=(eMe)\cup(fMf)$ [**7**, Theorem 5]. But H_e and H_f are degenerate so M is irreducible from e to f. Since eMe and fMf are acyclic plane clans, they are arcwise connected by Theorem 1. Then M is an arc from e to f, a contradiction. Thus M has a right (or left) identity e.

Then $Me=M$ and $eM=eMe$ is either degenerate or arcwise connected. If eM is degenerate, $e\in K$ and $Me=M=K$, a contradiction. Hence $eM=eMe$ is a nondegenerate arcwise connected clan with e as its identity. Let T be an arc in eM from e to its minimal ideal K' such that $T\cap K'$ is degenerate. Clearly $K'\subseteq K$. Since each of aT and bT is a continuous image of T, each is either degenerate or arcwise connected, and there is an α and a β such that each of α and β is an arc or degenerate, $\alpha\subseteq aT$, $\beta\subseteq bT$, α contains a, β contains b and each of α and β intersects K at only one point. Since M is irreducible from a to b, $M=\alpha\cup K\cup\beta$. If both a and b belong to K, $K=M$, so let $e\in\beta\backslash K$. If $e\neq b$, e possesses a euclidean (1-dimensional) neighborhood and since e is a right identity, $e\in K$, a contradiction [**2**, Lemma 4]. Hence $e=b$ and (3) holds.

REMARK. An application of Theorem 1 to some nonchainable continua would be as follows: no continuum in the plane consisting of an infinite half-ray "spiraling down" upon a nondegenerate acyclic continuum admits the structure of a topological semigroup with identity.

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