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A GENERAL THREE-SERIES THEOREM

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ABSTRACT. Let $\{\Omega, \mathfrak{F}, P\}$ be a probability space. The subset of Ω on which an arbitrary sequence of random variables converges is shown to be equivalent to the intersection of three other sets, each specified by the almost sure convergence of a certain sequence of random variables. Kolmogorov's three-series theorem, which gives necessary and sufficient conditions for the almost sure convergence of a sequence of sums of independent random variables, is obtainable as a particular case of the present result.

1. **Introduction.** Kolmogorov's famous three-series theorem states that the sums of a sequence of independent random variables (r.vs) converge if and only if three other series (of constants) also converge. In the present work, the initial assumption of independence is dispensed with and we obtain the result (Theorem 1) that sums of a sequence of arbitrary r.vs converge if and only if three other series (of r.vs) converge.

The r.vs of the three series in question are derived from the original r.vs by use of conditional expectation operators, and become constants when the original r.vs are independent. Thus, Theorem 1 contains the Kolmogorov theorem as a special case.

In the case of independent r.vs, the probability of convergence is clearly 0 or 1, but without independence, the probability of convergence may lie between 0 and 1. Theorem 1 may therefore be viewed as specifying the set on which convergence occurs as the intersection of three other sets, upon each of which a certain series of r.vs converges.

Theorem 1 is stated below. Its proof, given in §3, follows without difficulty from known results of Lévy, Doob and Burkholder, and from the three-series theorem of Kolmogorov. §2 contains a list of these and other preliminary results.

Let $\{X_n, \mathfrak{F}_n, n=0, 1, 2, \cdots\}$ be a stochastic sequence on the probability space $\{\Omega, \mathfrak{F}, P\}$, defined by taking $\mathfrak{F}_n = \mathfrak{B}(X_0, X_1, \cdots, X_n)$, the Borel σ-field generated by $X_0, X_1, \cdots, X_n$. Denote the indicator function of a set A by $I(A)$. Let $K < \infty$ be an arbitrary positive constant and write

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$$Y_n = X_n I(|X_n| \leqq K), \qquad n = 1, 2, \cdots.$$

In notation such as $\sum_j$, $\prod_j$ and $\bigcap_j$, the (countable) index j is always taken to run from 1 to ∞. The statement "$\sum_j Z_j$ converges" is used to denote $\sum_{j=1}^n Z_j$ converges as $n \to \infty$. We say that two sets are *equivalent* if their symmetric difference has probability zero.

THEOREM 1. *Let* $B = [\sum_j X_j$ *converges*]. *Then* B *is equivalent to the set on which*

$$\sum_j P[|X_j| > K \mid \mathfrak{F}_{j-1}] < \infty, \tag{1}$$

$$\sum_j E(Y_j \mid \mathfrak{F}_{j-1}) \text{ converges,} \tag{2}$$

and

$$\sum_j E((Y_j - E(Y_j \mid \mathfrak{F}_{j-1}))^2 \mid \mathfrak{F}_{j-1}) < \infty \tag{3}$$

all hold. Also, the set on which (3) *holds is equivalent to the set on which*

$$\sum_j (Y_j - E(Y_j \mid \mathfrak{F}_{j-1}))^2 < \infty. \tag{4}$$

2. **Preliminaries.** We list the results upon which the proof of the theorem depends.

KOLMOGOROV'S THREE-SERIES THEOREM. *If* $X_1, X_2, \cdots$ *are independent r.vs, then* $\sum_j X_j$ *converges a.s. if and only if*

$$\sum_j P[|X_j| > K] < \infty, \qquad \sum_j EY_j \text{ converges, and}$$

$$\sum_j E(Y_j - EY_j)^2 < \infty.$$

LEMMA 1 (BURKHOLDER, THEOREM 4 OF [2]). *Let* $\{V_n, \mathcal{G}_n, n = 1, 2, \cdots\}$ *be a martingale difference sequence for which* $E \sup_n |V_n| < \infty$. *Then the set* $[\sum_j V_j^2 < \infty]$ *is equivalent to the set on which* $\sum_j V_j$ *converges.*

LEMMA 2 (DOOB, COROLLARY 1, P. 323 OF [1]). *Let* $\{V_n, \mathcal{G}_n, n = 1, 2, \cdots\}$ *be a stochastic sequence for which*

$$P[0 \leqq V_n \leqq K] = 1, \qquad \text{all } n = 1, 2, \cdots,$$

for some positive finite K. *Then the two sets* $[\sum_j V_j < \infty]$ *and* $[\sum_j E(V_j \mid \mathcal{G}_{j-1}) < \infty]$ *are equivalent.*

As a corollary to Lemma 2 we obtain

LEMMA 3 (THE CONDITIONAL BOREL-CANTELLI LEMMA, DUE TO LÉVY). *Let $A_1, A_2, \cdots$ be a sequence of sets and $\mathcal{G}_0 \subset \mathcal{G}_1 \subset \mathcal{G}_2 \subset \cdots$ a sequence of σ-fields such that $A_n \in \mathcal{G}_n$, $n=1, 2, \cdots$. Then the two sets $[\sum_j I(A_j) < \infty]$ and $[\sum_j P(A_j | \mathcal{G}_{j-1}) < \infty]$ are equivalent.*

LEMMA 4. Let $\phi_1(\cdot), \phi_2(\cdot), \cdots$ *be a sequence of moment generating functions (m.g.fs) of distribution functions, with*

$$\phi_j(t) = \int_{-\infty}^{\infty} e^{tx} dF_j(x), \tag{5}$$

and

$$1 - F_j(x) + F_j(-x) = 0 \tag{6}$$

for all $x > K$, and all $j=1, 2, \cdots$. If there is a real function ψ with $\psi(t) > 0$ for all t such that

$$\lim_{n\to\infty} \prod_{j=1}^{n} \phi_j(t) = \psi(t) \qquad \textit{for all } t, \tag{7}$$

then $\sum_j \phi_j'(0)$ converges.

PROOF. By equations (5) and (6), ϕ_j is the m.g.f. of a r.v. U_j for which $P[|U_j| \leq K] = 1$, all $j=1, 2, \cdots$. From equation (7), for each fixed t, $\lim_{n\to\infty} \prod_{j=n+1}^{n+m} \phi_j(t) = 1$ uniformly in $m=1, 2, \cdots$. Thus, if $U_1, U_2, \cdots$ are *independent* with $S_n = U_1 + \cdots + U_n$, then

$$\lim_{n\to\infty} E \exp(t(S_{n+m} - S_n)) = 1 \tag{8}$$

for fixed t, uniformly in m. Therefore (8) holds uniformly in m when t is replaced by $-t$. Adding, we obtain

$$\lim_{n\to\infty} E \cosh(t(S_{n+m} - S_n)) = 1$$

for fixed t, uniformly in m. But $\cosh x$ is convex with a minimum of 1 at $x=0$, so that $S_{n+m} - S_n \to 0$ in probability as $n \to \infty$, uniformly in m, i.e. there exists a r.v. S such that $S_n \to S$ in probability as $n \to \infty$. Therefore $\lim_{n\to\infty} S_n = S$ a.s., since S_n is the sum of independent r.vs. Then, by Kolmogorov's three-series theorem, $\sum_j EU_j = \sum_j \phi_j'(0)$ converges.

3. **Proof of the theorem.** The equivalence of the sets on which (3) or (4) hold follows from Lemma 2 and the uniform boundedness of the $\{Y_n\}$. It then suffices to show that B is equivalent to the set D on which (1), (2) and (4) hold.

(i) *Sufficiency*. By (4) and Lemma 1,

(9) $$\sum_j (Y_j - E(Y_j \mid \mathfrak{F}_{j-1})) \quad \text{converges a.s. on } D.$$

Also, from (1) and Lemma 3,

(10) $$\sum_j I(X_j \neq Y_j) < \infty \quad \text{a.s. on } D.$$

(9) and (10) together with (2) show that $\sum_j X_j$ converges a.s. on D, i.e. that $P(D-B)=0$.

(ii) *Necessity*. To show that $P(B-D)=0$, we note first that (10) holds, and hence, by Lemma 3, (1) holds, a.s. on B. Thus it remains to show that $P(B-D')=0$, where D' is the set on which (2) and (4) hold. To do this, we introduce the set B' on which

(11) $$\sum_j Y_j \text{ converges,}$$

and show that $P(B-B')=P(B'-D')=0$. Because (10) holds a.s. on B, (11) also holds a.s. on B, i.e. $P(B-B')=0$. Therefore it remains to show that $P(B'-D')=0$, i.e. that (2) and (4) each hold a.s. on B', and to do this it suffices, in turn, to show that (2) holds a.s. on B', for then (9) holds a.s. on B' and hence, by Lemma 1, (4) holds a.s. on B'.

Let $T_n = Y_1 + \cdots + Y_n$, and for fixed real θ let

$$W_n(\theta) = e^{\theta T_n}\left(\prod_{j=1}^{n} E(e^{\theta Y_j} \mid \mathfrak{F}_{j-1})\right)^{-1}.$$

With $W_0(0)=1$, $\{W_n(\theta), \mathfrak{F}_n, n=1, 2, \cdots\}$ is a martingale. Moreover $W_n(\theta) \geqq 0$ and $EW_n(\theta)=1$ for all $n=1, 2, \cdots$, so that $W_n(\theta)$ converges a.s. as $n\to\infty$ by the martingale convergence theorem. But T_n, and hence $e^{\theta T_n}$, converges a.s. as $n\to\infty$ on B'. Therefore

(12) $$\prod_j E(e^{\theta Y_j} \mid \mathfrak{F}_{j-1}) \text{ converges}$$

to a nonzero limit for $\omega \in M$, where $P(B'-M)=0$.

For each j and almost all fixed $\omega \in \Omega$, the r.v.

(13) $$\phi_j(\theta) = E(e^{\theta Y_j} \mid \mathfrak{F}_{j-1})$$

is the m.g.f. of a uniformly bounded r.v. Specifically, there exist sets $M_1, M_2, \cdots$ such that $M_j \in \mathfrak{F}_{j-1}$, $PM_j=1$, and equations (5) and (6) hold for all $\omega \in M_j$, $j=1, 2, \cdots$; thus

(14) $$F_j(x) = E(I(Y_j \leqq x) \mid \mathfrak{F}_{j-1}) \quad \text{for } \omega \in M_j.$$

(For further amplification see the Remark below.) By (12) and Lemma 4, $\sum_j \phi_j'(0)$ converges for $\omega \in (\bigcap_j M_j) \cap M$. But $\phi_j'(0) = E(Y_j \mid \mathfrak{F}_{j-1})$ for all ω. Also $P(B' - (\bigcap_j M_j) \cap M) = 0$ since $P(B' - M) = 0$ and $PM_j = 1$ for all j. Therefore (2) holds a.s. on B', completing the proof of necessity, and of the theorem.

REMARK. Because conditional expectations are defined uniquely only up to sets of probability zero, it should be verified that, with $\phi_j(\theta)$ and $F_j(x)$ as given by (13) and (14), equation (5) and Lemma 4 do not depend upon an uncountable range of values of θ and/or x. In fact Lemma 4 depends only on equation (5) holding for two values of θ, while the required version of (5) itself may be written

$$\phi_j(\theta) = \lim_{n \to \infty} \sum_{k=-2^n}^{2^n} e^{\theta (k-1) 2^{-n}} P[K \cdot (k-1) 2^{-n} < Y_j \leqq K k 2^{-n} \mid \mathfrak{F}_{j-1}]$$

(by the monotone convergence theorem for conditional expectations), thus involving only a countable number of conditional expectations.

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