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# p.p. RINGS AND FINITELY GENERATED FLAT IDEALS

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ABSTRACT. In this note all rings considered are associative with an identity element 1 and all modules are unital left modules. It is shown that a commutative ring $R$ has principal ideals projective if and only if $R[X]$ has the same property. Furthermore it is proved that a ring $R$ has all $n$-generated left ideals flat if and only if all $n$-generated right ideals are flat. In the last part of this note we will prove the following results:

Fix $n \geq 1$. Then there exists a ring $R$ such that all $n$-generated left ideals are projective, in particular, flat, while there exists a nonflat $(n+1)$-generated left ideal.

1. **Stabilization results for p.p. rings.** Rings which have principal left ideals projective are called left p.p. rings and rings which have principal left ideals flat are called left p.f. rings.

For sake of completeness let us state the following well-known lemma:

LEMMA 1.1. *Let $R$ be a commutative ring and $M$ a finitely generated $R$-module. Then the following statements are equivalent*:

(1) *$M$ is $R$-projective.*

(2) *$M_P$ is $R_P$-free for all $P \in \operatorname{Spec}(R)$ and the function $P \to \operatorname{rank}_{R_P}(M_P)$ is a continuous function from $\operatorname{Spec}(R)$ to the integers $Z$.*

THEOREM 1.2. *Let $R$ be a commutative ring. $R$ is a p.p. ring if and only if $R[X]$ is a p.p. ring.*

PROOF. It is easy to prove that if $R[X]$ is a p.p. ring, then $R$ is a p.p. ring. Assume now that $R$ is a p.p. ring and let $R[X](a_0 + \cdots + a_m X^m)$ be any principal ideal in $R[X]$. For $P \in \operatorname{Spec}(R[X])$ we have that $(a_0)_P \neq (0)_P$ or $(a_0)_P = (0)_P$. Let us first consider the case where $(a_0)_P \neq (0)_P$. Since $Ra_0$ is $R$-projective it follows that $R[X]a_0$ is $R[X]$-projective and hence, by Lemma 1.1, $(R[X]a_0)_P$ is $(R[X])_P$-free of rank one. This means that $(a_0)_P$ is a nonzero divisor in $(R[X])_P$, i.e. $(R[X](a_0 + \cdots + a_m X^m))_P$ is free of rank one.

If $(a_0)_P = (0)_P$, then there exists an element $c(X) \notin P$ such that $a_0 c(X) = 0$.

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$$(R[X](a_0 + \cdots + a_m X^m))_P = (R[X])_P \left( \frac{(a_0 + \cdots + a_m X^m) c(X)}{c(X)} \right)_P$$
$$= (R[X])_P (a_1 + \cdots + a_m X^m)_P$$
$$\simeq (R[X](a_1 + \cdots + a_m X^{m-1}))_P.$$

By repeating this argument we get that for all $P \in \operatorname{Spec}(R[X])$ $(R[X](a_0 + \cdots + a_m X^m))_P$ is free of rank one or rank zero. Furthermore the argument shows that

$$\operatorname{Supp}(R[X](a_0 + \cdots + a_m X^m)) = \bigcup_{i=0}^{m} \operatorname{Supp}_{R[X]}(a_i).$$

Theorem 1.2 follows now from Lemma 1.1.

REMARK. The argument, which proved Theorem 1.2, also proves that a ring $R$ is a p.f. ring if and only if $R[X]$ is a p.f. ring.

EXAMPLE. Let $R$ be $Z_2$ (the ring of $2 \times 2$ matrices over the integers). $R$ is a left and right p.p. ring, $R[X]\begin{pmatrix} 2 & 0 \\ X & 0 \end{pmatrix}$ is a nonprojective principal left ideal in $R[X]$ (the annihilator of $\begin{pmatrix} 2 & 0 \\ X & 0 \end{pmatrix}$ cannot be generated by an idempotent matrix).

This example is essentially due to P. M. Cohn.

2. **p.p. rings and p.f. rings.** While it is not true that a right p.p. ring is a left p.p. ring (cf. S. U. Chase [**3**]), we can, however, prove that a right p.f. ring is a left p.f. ring. But first we need a lemma.

LEMMA 2.1. *Let $I$ be the left ideal generated by $(a_1, \cdots, a_m)$. Then $I$ is a flat left ideal if and only if for all $(b_1, \cdots, b_m)$ with $b_1 a_1 + \cdots + b_m a_m = 0$, we can find an $m \times m$ matrix $\boldsymbol{A}$ such that the following conditions are satisfied*:

$$(1) \qquad (b_1, \cdots, b_m)\boldsymbol{A} = (b_1, \cdots, b_m),$$

$$(2) \qquad \boldsymbol{A} \begin{pmatrix} a_1 \\ \cdot \\ \cdot \\ \cdot \\ a_m \end{pmatrix} = \begin{pmatrix} 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{pmatrix}.$$

PROOF. The lemma might follow from [**2**, Proposition 2.3]. Let us consider the exact sequence of left $R$-modules

$$0 \to K \to F \xrightarrow{g} I \to 0,$$

where $F$ is free with base $(e_1, \cdots, e_m)$ and $g(e_j) = a_j$ for all $j \in \{1, \cdots, m\}$. $K$ denotes the kernel of $g$.

It is well known that $I$ is flat if and only if for $k \in K$ there exists a homomorphism $u \in \operatorname{Hom}(F, K)$ such that $u(k) = k$. Any $k \in F$ is equal to $b_1e_1+ \cdots +b_me_m$ for some $b_j$'s in the ring $R$. We see that $k\in K$ if and only if $b_1a_1+ \cdots +b_ma_m=0$. The existence of a homomorphism $u\in \mathrm{Hom}(F, K)$ such that $u(k)=k$ is equivalent to the existence of an $m\times m$ matrix $\boldsymbol{A}$ such that

$$\boldsymbol{A}\begin{pmatrix} a_1 \\ \cdot \\ \cdot \\ \cdot \\ a_m \end{pmatrix} = \begin{pmatrix} 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{pmatrix} \quad \text{and} \quad (b_1, \cdots, b_m)\boldsymbol{A} = (b_1, \cdots, b_m).$$

This completes the proof of the lemma.

THEOREM 2.2. *If all $n$-generated left ideals are flat, then all $n$-generated right ideals are flat, too.*

PROOF. By the "dual" to Lemma 2.1 it follows that all $n$-generated right ideals are flat if and only if for any equation

$$b_1a_1 + \cdots + b_na_n = 0$$

there exists an $n\times n$ matrix $\boldsymbol{B}$ such that

$$\boldsymbol{B}\begin{pmatrix} a_1 \\ \cdot \\ \cdot \\ \cdot \\ a_n \end{pmatrix} = \begin{pmatrix} a_1 \\ \cdot \\ \cdot \\ \cdot \\ a_n \end{pmatrix} \quad \text{and} \quad (b_1, \cdots, b_n)\boldsymbol{B} = (0, \cdots, 0).$$

If we have an equation $b_1a_1+ \cdots +b_na_n=0$ then by Lemma 2.1 we can find an $n\times n$ matrix $\boldsymbol{A}$ such that

$$(b_1, \cdots, b_n)\boldsymbol{A} = (b_1, \cdots, b_n) \quad \text{and} \quad \boldsymbol{A}\begin{pmatrix} a_1 \\ \cdot \\ \cdot \\ \cdot \\ a_n \end{pmatrix} = \begin{pmatrix} 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{pmatrix}.$$

It is now obvious that the matrix $\boldsymbol{E}-\boldsymbol{A}$ can be used as the matrix $\boldsymbol{B}$ (here $\boldsymbol{E}$ denotes the $n\times n$ identity matrix).

COROLLARY 1. *If $R$ is a left p.p. ring, then any principal right ideal is flat and for any $a\in R$ we have that the right annihilator of $a$, $r(a)$, is generated by idempotents $(e_i)_{i\in I}$, with the property that for all $i_1$ and $i_2$, where $i_1$ and $i_2$ are elements in $I$, we can find an $i\in I$ such that $e_{i_1}R \subseteq e_iR$ and $e_{i_2}R\subseteq e_iR$.*

PROOF. The first statement follows from Theorem 2.2 and the second from the proof of Theorem 2.2. The last statement is a consequence of [**1**, Chapter 1, §2, Exercise 23b].

COROLLARY 2 (L. W. SMALL [**7**]). *Let $R$ be a ring in which every principal left ideal is projective and in which there is no infinite set of* *nonzero orthogonal idempotents. Then every principal right ideal is projective.*

COROLLARY 3 (S. ENDO [5]). *Assume that all idempotents in the ring $R$ are central. $R$ is a left p.p. ring if and only if $R$ is a right p.p. ring.*

EXAMPLE. (Found jointly with P. M. Cohn.) Let $n$ be any natural number and let $R$ be the $K$-algebra ($K$ is any commutative field) on the $2(n+1)$ generators $X_i$, $Y_i$ $(i=1, \cdots, n+1)$ and defining relations

$$\sum_{i=1}^{n+1} X_i Y_i = 0. \tag{3}$$

It follows from [4, Theorem 2.3] and [2, Theorem 3.1] that the ring $R$ is an *n-fir*. We want to prove that there exists a nonflat $(n+1)$-generated left ideal. If we assume that all $(n+1)$-generated left ideals are flat, then the relation (3) will imply that there exists a matrix $\boldsymbol{A}$ ($\boldsymbol{A}$ is an $(n+1)\times(n+1)$ matrix) such that

$$(X_1, \cdots, X_{n+1})\boldsymbol{A} = (X_1, \cdots, X_{n+1}) \quad \text{and} \quad \boldsymbol{A}\begin{pmatrix} Y_1 \\ \cdot \\ \cdot \\ \cdot \\ Y_{n+1}\end{pmatrix} = \begin{pmatrix} 0 \\ \cdot \\ \cdot \\ \cdot \\ 0\end{pmatrix}.$$

The matrix $\boldsymbol{A}$ can be written as a sum of two matrices $\boldsymbol{A}_0$ and $\boldsymbol{A}_1$, where $\boldsymbol{A}_0$ is a scalar matrix and $\boldsymbol{A}_1=(a_{ij})$, where $a_{ij}$ has no nonzero scalar terms. It is easy to see that

$$(X_1, \cdots, X_{n+1})\boldsymbol{A}_0 = (X_1, \cdots, X_{n+1}) \quad \text{and} \quad \boldsymbol{A}_0\begin{pmatrix} Y_1 \\ \cdot \\ \cdot \\ \cdot \\ Y_{n+1}\end{pmatrix} = \begin{pmatrix} 0 \\ \cdot \\ \cdot \\ \cdot \\ 0\end{pmatrix}.$$

If we specialize $X_i \to 0$ and $Y_i \to Y_i$, we see that there are no relations between the $Y_i$'s, hence we get that $\boldsymbol{A}_0=\boldsymbol{0}$, a contradiction. We have now proved the following:

THEOREM 2.3. *For every $n \geqq 1$, there exists a ring $R$ such that any $n$-generated left ideal is flat, while there exists a nonflat $(n+1)$-generated left ideal. We might even choose $R$ to be an n-fir.*

REMARK. For commutative rings we have that all ideals are flat if the ideals generated by two elements are flat [6, Lemma 4.1], and a commutative ring $R$ is semihereditary if all ideals generated by two elements are projective [6, Proposition 4.2] or [8, Theorem 4.2].

## References

1. N. Bourbaki, *Algèbre commutative*. Chaps. 1, 2, fasc. 27, Actualités Sci. Indust., no. 1290, Hermann, Paris, 1961. MR **36** #146.

2. S. U. Chase, *Direct products of modules*, Trans. Amer. Math. Soc. **97** (1960), 457–473. MR **22** #11017.

3. ———, *A generalization of the ring of triangular matrices*, Nagoya Math. J. **18** (1961), 13–25. MR **23** #A919.

4. P. M. Cohn, *Dependence in rings*. II. *The dependence number*, Trans. Amer. Math. Soc. **135** (1969), 267–279.

5. S. Endo, *Note on p.p. rings*, Nagoya Math. J. **17** (1960), 167–170. MR **25** #1195.

6. S. Jøndrup, *On finitely generated flat modules*. II, Math. Scand. **27** (1970), 105–112.

7. L. W. Small, *Semihereditary rings*, Bull. Amer. Math. Soc. **73** (1967), 656–658. MR **35** #2926.

8. W. V. Vasconcelos, *On finitely generated flat modules*, Trans. Amer. Math. Soc. **138** (1969), 505–512. MR **39** #199.

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