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ON THE CONJUGACY OF INJECTORS

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ABSTRACT. In their paper, *Injektoren endlicher auflösbarer Gruppen*, Fischer, Gaschütz and Hartley ask the following question. If $\mathfrak{F}$ is a normal subgroup closed class of groups and if G is a finite solvable group which possesses $\mathfrak{F}$-injectors, is it true that any two $\mathfrak{F}$-injectors of G are conjugate in G? A partial answer is given. It is proven that if G has p-length 1 for each prime p, then the answer to this question is yes.

1. **Introduction.** Fitting classes and injectors were introduced by Fischer, Gaschütz and Hartley [2]. A Fitting class $\mathfrak{F}$ is an isomorphism closed class of groups satisfying $f_1: G \in \mathfrak{F}$, $N \lhd G$ implies $N \in \mathfrak{F}$, $f_2: N_1$, $N_2 \lhd G$, N_1, $N_2 \in \mathfrak{F}$ implies $N_1N_2 \in \mathfrak{F}$. If G is a group, $V \leqq G$ is an $\mathfrak{F}$-injector of G provided $N \lhd\lhd G$ implies $V \cap N$ is $\mathfrak{F}$-maximal in N. Satz 1 [2] states that if $\mathfrak{F}$ is a Fitting class and G is a finite solvable group, then G possesses $\mathfrak{F}$-injectors and any two are conjugate. At the close of [2] the authors ask if the conjugacy of injectors can be proven using only the first of the defining properties of a Fitting class. That is, if $\mathfrak{F}$ is an isomorphism closed class of groups satisfying f_1 and if G is a finite solvable group which possesses $\mathfrak{F}$-injectors, is it true that any two $\mathfrak{F}$-injectors of G are conjugate? A partial answer is given. We prove that if G has p-length 1 for each prime p, then the answer to this question is yes.

2. **p-normally embedded subgroups.** In proving our result we will use the concept of a p-normally embedded subgroup. $V \leqq G$ is said to be p-normally embedded in G if a Sylow p-subgroup V_p of V is also Sylow in some normal subgroup of G. This concept was introduced by Hartley [3] and has also been studied in [1]. We are going to need the following theorem which is essentially a restatement of Theorem 2.6 of [1].

THEOREM 1. *Let G be a finite solvable group and suppose $V \leqq G$ is p-normally embedded in G for each prime p. Suppose $W \leqq G$ and that for each prime p the Sylow p-subgroups of W are conjugate to those of V. Then V and W are conjugate.*

We are also going to need the following theorem which will be used

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to show that if G has p-length 1 for each prime p, then the $\mathfrak{F}$-injectors of G are p-normally embedded in G.

THEOREM 2. *Let p be a prime and let G be a p-solvable finite group. Then G has p-length* 1 *if and only if each p-subgroup of G is Sylow in some subnormal subgroup of G.*

PROOF. Suppose G has p-length 1 and that P is a p-subgroup of G. Let $K=O_{p'}(G)$ and consider G/K. PK/K is a p-subgroup of G/K and, if $K\neq 1$, PK/K is Sylow in some $L/K\lhd\lhd G/K$ by induction. But then P is Sylow in $L\lhd\lhd G$ as required. Thus we may assume $K=1$. Then G has a normal Sylow p-subgroup P^* and $P\lhd\lhd P^*\lhd G$ so that $P\lhd\lhd G$.

To prove the converse we suppose each p-subgroup of G is Sylow in some subnormal subgroup of G. If $N\lhd G$ and P/N is a p-subgroup of G/N, then there is a p-subgroup P^* of G such that $P=P^*N$. By assumption P^* is Sylow in some $L\lhd\lhd G$ so that $P/N=P^*N/N$ is Sylow in $LN/N\lhd\lhd G/N$. Thus by induction G/N has p-length 1 for any $1\neq N\lhd G$. If $O_{p'}(G)\neq 1$, then we are done. Otherwise we can assume G has a unique minimal normal subgroup K which is a p-group. If $\Phi(G)\neq 1$, then $G/\Phi(G)$ has p-length 1 and hence so does G. Thus we may assume $\Phi(G)=1$ so that K is complemented. Assume $MK=G$ and $M\cap K=1$. If M is p', then K is Sylow p in G and we are done. Suppose then that $1\neq M_p$ is Sylow p in M. By assumption M_p is also Sylow in some $L\lhd\lhd G$. Since K is a p-group and $M_p\cap K=1$, L is a proper subgroup of G. But then there exists a proper normal subgroup L^* of G such that $M_p\leqq L\leqq L^*$. Since K is the unique minimal normal subgroup of G, $K\leqq L^*$. Then $M_pK\leqq L^*$ so that L^* has p' index. Now each p-subgroup of L^* is Sylow in some $R\lhd\lhd G$ and so is Sylow in $L^*\cap R\lhd\lhd L^*$. Thus L^* has p-length 1 by induction. Since L^* has p' index this implies G has p-length 1. Q.E.D.

3. **The main theorem.**

THEOREM 3. *Suppose G has p-length* 1 *for each prime p and suppose V and W are $\mathfrak{F}$-injectors of G where $\mathfrak{F}$ is an isomorphism closed class of groups satisfying f_1. Then*

(1) *V is p-normally embedded in G for each prime p.*

(2) *V and W are conjugate.*

PROOF. The proof is by induction on $|G|$. We assume both statements have been shown to hold whenever $|G|<n$. Now assume $|G|=n$. Our first step is to show that $|V|=|W|$. Let M be a maximal normal subgroup of G. $V\cap M$ and $W\cap M$ are each $\mathfrak{F}$-injectors of

M and so they are conjugate by induction. Suppose in fact that $V^g \cap M = (V\cap M)^g = W \cap M$. If $W = W\cap M$, then $W \leqq V^g$ and, since W and V^g are each $\mathfrak{F}$-injectors, this would imply $W = V^g$. Certainly $|W| = |V|$ in this case. Thus we may assume $WM = G$ and $|W| = [G:M]|W\cap M|$. Similarly we may assume $|V| = [G:M]|V\cap M|$ and once again we have $|W| = |V|$.

Let V_p and W_p denote Sylow p-subgroups of V and W respectively. Our second step is to show that V_p and W_p are conjugate. If both V_p and W_p are Sylow in G, this is clear. Suppose then that V_p is not Sylow in G. From Theorem 2 we know V_p is Sylow in some proper subnormal subgroup L of G. $V\cap L$ and $W\cap L$ are each $\mathfrak{F}$-injectors of L and so they are conjugate by induction. Choose g such that $V\cap L = (W\cap L)^g \leqq W^g$. Then V_p is Sylow in $V\cap L \leqq W^g$ so that V_p is contained in some conjugate of W_p. Since V and W have the same order so do V_p and W_p and so we conclude that V_p and W_p are conjugate.

The next step is to show that V is p-normally embedded in G. By Theorem 2, V_p is Sylow in some $L \triangleleft\triangleleft G$. If $L = G$, then V_p is Sylow in G so that V is p-normally embedded in G. If L is proper then there is a proper normal subgroup H of G such that $V_p \leqq L \leqq H$. $V\cap H$ is an $\mathfrak{F}$-injector of H and V_p is Sylow in $V\cap H$. Since $H < G$, $V\cap H$ is p-normally embedded in H by induction. That is, V_p is Sylow in some normal subgroup K of H. But then V_p is Sylow in $(V_p)^H \leqq K$. Suppose now that $\alpha \in \mathrm{Aut}(H)$. Then $(V\cap H)^\alpha$ is again an $\mathfrak{F}$-injector of H and since $|H| < |G|$, $(V\cap H)^\alpha$ is conjugate to $V\cap H$ in H by induction. In particular $(V_p)^\alpha$ is conjugate to V_p in H. This shows that $(V_p)^H$ is in fact characteristic in $H \triangleleft G$. But then V_p is Sylow in $(V_p)^H \triangleleft G$ so that V is p-normally embedded in G as required.

As a final step we invoke Theorem 1 to complete the proof that V and W are conjugate. Q.E.D.

References

1. G. A. Chambers, *p-normally embedded subgroups of finite soluble groups*, J. Algebra **16** (1970), 442–455.

2. B. Fischer, W. Gaschütz and B. Hartley, *Injektoren endlicher auflösbarer Gruppen*, Math. Z. **102** (1967), 337–339. MR **36** #6504.

3. B. Hartley, *On Fischer's dualization of formation theory*, Proc. London Math. Soc. (3) **19** (1969), 193–207. MR **39** #5696.

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