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FUNCTIONS WHICH ARE FOURIER-STIELTJES TRANSFORMS

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ABSTRACT. Let G be a locally compact abelian group, $\hat{G}$ the dual group, $M(G)$ the algebra of regular bounded Borel measures on G, and $M(G)^\wedge$ the algebra of Fourier-Stieltjes transforms. The purpose of this paper is to characterize those continuous functions on $\hat{G}$ which belongs to $M(X)^\wedge$, where X is a closed subset of G and $M(X)=\{\mu\in M(G)$: the support of μ is contained in $X\}$.

More precisely, we will prove the following theorem:

THEOREM. *Let X be a closed subset of G and f a continuous function on $\hat{G}$. Then the following are equivalent:*

(a) $f\in M(X)^\wedge$.

(b) $\{\lambda_n\}\subset M(\hat{G})$, $|\check{\lambda}_n(x)|\leqq M$ *for all* $x\in X$ *and* $\check{\lambda}_n(x)\to 0$ *for all* $x\in X$ *implies* $\int_{\hat{G}} f d\lambda_n\to 0$.

The case where f is assumed bounded and $X=G$ was proved by Ramirez in [2] by applying Grothendieck's completion theorem [1, p. 271] to the paired spaces $M(G)^\wedge$ and $M(\hat{G})$ under the pairing $(\hat{\mu},\lambda)=\int_{\hat{G}}\hat{\mu}d\lambda$ where $\mu\in M(G)$ and $\lambda\in M(\hat{G})$.

We provide a short proof of the more general result using the well-known theorem of Eberlein (see, for example [3, p. 32]), which states that a continuous function f on $\hat{G}$ is a Fourier-Stieltjes transform if and only if there exists a constant A such that

$$\left|\sum_{i=1}^{n} c_i f(\gamma_i)\right| \leqq A\|p\|_\infty, \qquad \gamma_i\in\hat{G},$$

for every trigonometric polynomial p on G of the form

$$p(x)=\sum_{i=1}^{n} c_i\gamma_i(x), \qquad x\in G.$$

PROOF. Suppose $f=\hat{\mu}$ where $\mu\in M(X)$ and $\{\lambda_n\}$ satisfies the hypotheses of (b). Then by Fubini's theorem and the Lebesgue dominated convergence theorem, we have

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$$\int_{\hat{G}} f d\lambda_n = \int_X \hat{\lambda}_n d\mu \to 0.$$

Now assume that f satisfies (b). We will first show that $f \in M(G)\hat{}$. By Eberlein's theorem we must show that if $\{p_n\}$ is a sequence of trigonometric polynomials on G, say $p_n(x) = \sum_{i=1}^{k(n)} c_{in}\gamma_{in}(x)$ where $x \in G$ and $\gamma_{in} \in \hat{G}$, with $p_n \to 0$ uniformly on G, then $\sum_{i=1}^{k(n)} c_{in} f(\gamma_{in}) \to 0$ as $n \to \infty$.

Now let $\lambda_n = \sum_{i=1}^{k(n)} c_{in}\delta_{in} \in M(\hat{G})$ where δ_{in} is the point mass at γ_{in}. Since $\hat{\lambda}_n = p_n$ for every n, we have that $\{\lambda_n\}$ satisfies the hypotheses of (b) and hence

$$\sum_{i=1}^{k(n)} c_{in} f(\gamma_{in}) = \int_{\hat{G}} f d\lambda_n \to 0 \quad \text{as } n \to \infty.$$

Therefore $f = \hat{\mu}$ where $\mu \in M(G)$ and hence we need only show that the support of μ is contained in X. Suppose that this is not the case. Then the regularity of μ allows us to choose a compact set $E \subset X'$, the complement of X, such that $\mu(E) \neq 0$ and a sequence $\{U_n\}$ of open sets satisfying $E \subset U_n \subset X'$ and $|\mu|(U_n \backslash E) < n^{-1}$ for every n.

Now choose a sequence $\{\lambda_n\} \subset M(\hat{G})$ with $0 \leqq \hat{\lambda}_n \leqq 1$, $\hat{\lambda}_n = 0$ outside U_n, and $\hat{\lambda}_n = 1$ on E for every n. Clearly $\{\lambda_n\}$ satisfies the hypotheses of (b) and hence $\int_{\hat{G}} f d\lambda_n \to 0$.

However,

$$\left| \int_{\hat{G}} f d\lambda_n \right| = \left| \int_{U_n} \hat{\lambda}_n d\mu \right| \geqq \left| \int_E \hat{\lambda}_n d\mu \right| - \left| \int_{U_n \backslash E} \hat{\lambda}_n d\mu \right|$$
$$\geqq |\mu(E)| - n^{-1} \to |\mu(E)| \neq 0.$$

But this is a contradiction.

REMARK. It should be noted that if the assumption of continuity is dropped and X is replaced by G, then (b) will imply that $f \in M(\overline{G})\hat{}$, where $\overline{G}$ is the Bohr compactification of G.

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