ON ORDERED POLYCYCLIC GROUPS

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ABSTRACT. It has been asserted that any (full) order on a torsion-free, finitely generated, nilpotent group is defined by some F -basis of G and that the group of σ -automorphisms of such a group is itself a group of the same kind. Examples provided herein demonstrate that both of these assertions are false; however, it is proved that the group of σ -automorphisms of an ordered, polycyclic group is nilpotent by abelian, and polycyclic.

1. Introduction. It is well known that if G is a torsion-free, finitely generated, nilpotent group, then G possesses a central series $\{1\} = F_0 \subseteq F_1 \subseteq \cdots \subseteq F_m = G$ such that F_{i+1}/F_i is an infinite cyclic group, $i = 0, 1, \dots, m-1$. Ree (see [5] and [6]) calls such a series an " F -series of G ." It is also clear that if $F_{i+1}/F_i = \langle x_{i+1}F_i \rangle$, where $x_{i+1} \in F_{i+1}$ for $i = 0, 1, \dots, m-1$, then each element g of G can be written uniquely in the form $g = x_1^{e_1} x_2^{e_2} \cdots x_m^{e_m}$, where $e_1, e_2, \dots, e_m$ are integers. Ree [6] calls the elements $x_1, x_2, \dots, x_m$ an " F -basis of G ." It follows easily that the torsion-free, finitely generated, nilpotent group G can be ordered lexicographically with respect to $e_1, \dots, e_m$ as follows: For $g_1 = x_1^{e_1} x_2^{e_2} \cdots x_m^{e_m}$, $g_2 = x_1^{f_1} x_2^{f_2} \cdots x_m^{f_m}$, we put $g_1 \leq g_2$ if and only if $e_i = f_i$ for $i = 1, 2, \dots, m$, or $e_i < f_i$ for some t such that $1 \leq t \leq m$ and $e_i = f_i$ for $i = 1, 2, \dots, t-1$. This lexicographic order on G is said, by Ree [5], to be "defined" by the F -basis $x_1, x_2, \dots, x_m$.

These concepts are used by Ree in [6], where it is asserted that any (full) order on a torsion-free, finitely generated, nilpotent group is defined by some F -basis of G , and in the proof of Theorem 2 [6], which asserts that the group of σ -automorphisms of a torsion-free, finitely generated, nilpotent group is itself a torsion-free, finitely generated, nilpotent group. Both of these assertions, as shown by Examples 1 and 2 of this note, are false. While it is not possible to establish a result as strong as the one suggested by Theorem 2 of [6], we can prove

THEOREM. *If G is an ordered, polycyclic group and if Δ denotes the group of σ -automorphisms of G , then Δ is nilpotent by abelian, and, moreover, Δ is polycyclic.*

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2. Definitions and notations. If G is a group on which there can be defined a (full) order relation $\leq$ with the property that $a, b, x, y \in G$ and $a \leq b$ imply $xay \leq xby$, then G is said to be an *ordered group* and $\leq$ is said to be an *order* on G . Associated with an order $\leq$ on G is the *positive cone* $P(G)$ of G , $P(G) = \{x | x \in G \text{ and } 1 \leq x\}$. It follows that the subset $P(G)$ of the ordered group G has the following properties:

- (i) $P(G) \cap P^{-1}(G) = \{1\}$;
- (ii) $P(G)P(G) \subseteq P(G)$;
- (iii) $x^{-1}P(G)x \subseteq P(G)$; and
- (iv) $P(G) \cup P^{-1}(G) = G$.

Conversely, if G is a group which possesses a subset $P(G)$ with properties (i)–(iv), then G is an ordered group with respect to the relation $\leq$ given by

$$a \leq b \text{ if and only if } a^{-1}b \in P(G).$$

A subgroup C of a group G ordered with respect to $\leq$ is *convex* (with respect to $\leq$) if $g \in G, c \in C$, and $1 \leq g \leq c$ imply $g \in C$.

If $D \subset C$ are convex subgroups of the ordered group G with the property that no convex subgroup of G lies strictly between D and C , then $D < C$ is a *jump* in the chain of convex subgroups of G .

If G and H are ordered groups and f is a mapping of G into H , then f is an *o-homomorphism* of G into H if f is a group homomorphism of G into H and f is order-preserving in the sense that $a, b \in G$ and $a \leq_1 b$ imply $f(a) \leq_2 f(b)$, where $\leq_1$ and $\leq_2$ denote the orders on G and H , respectively. Furthermore, if f is a one-to-one *o-homomorphism* of G onto H and if f^{-1} is an *o-homomorphism* of H onto G , then f is an *o-isomorphism* of G onto H . An *o-isomorphism* of an ordered group G onto G is an *o-automorphism* of G .

If G is a group, then the series $\{1\} = A_0 \subseteq A_1 \subseteq \cdots \subseteq A_n = G$ is a *cyclic normal (invariant) series* of G if A_i is a normal subgroup of $A_{i+1}(G)$ and A_{i+1}/A_i is cyclic, $i = 0, 1, \dots, n-1$. A group G is *polycyclic* if G possesses a cyclic normal series. Finally, by the *length* of a polycyclic group G , we mean the number of infinite cyclic factors A_{i+1}/A_i in any cyclic normal series of G . It is well known that the length of a polycyclic group is an invariant for that group.

3. Proofs.

EXAMPLE 1. Let G be the subgroup of the additive group of reals which is generated by $\{1, \sqrt{2}\}$; i.e., $G = \langle 1 \rangle + \langle \sqrt{2} \rangle$. By restricting the natural Archimedean order on the reals to G , G is an Archimedean ordered, finitely generated, abelian group, whence G possesses no proper, nontrivial, convex subgroups.

Let $\{1\} = F_0 \subseteq F_1 \subseteq \cdots \subseteq F_m = G$ be an F -series of G with corre-

sponding F -basis $f_1, f_2, \dots, f_m$. Then, as G is polycyclic of length two, $m=2$, so that $\{1\} = F_0 \subset F_1 \subset F_2 = G$, $F_1 = \langle f_1 \rangle$ and $F_2/F_1 = \langle f_2 F_1 \rangle$. Thus, each element $g \in G$ can be written uniquely in the form $e_1 f_1 + e_2 f_2 = g$, where e_1, e_2 are integers. Let $\leq_1$ denote the order on G defined by the F -basis f_1, f_2 .

Now suppose that $g = e_1 f_1 + e_2 f_2$ and $0 \leq_1 e_1 f_1 + e_2 f_2 \leq_1 e'_2 f_2$, where e_1, e_2 , and e'_2 are integers. Then $0 \leq_1 -e_1 f_1 + (e'_2 - e_2) f_2$. Thus, as $0 \leq_1 e_1$ and $0 \leq_1 -e_1$, $e_1 = 0$ and, hence, $g = e_2 f_2 \in \langle f_2 \rangle$. Therefore, $\langle f_2 \rangle$ is a proper, nontrivial, convex subgroup of G with respect to $\leq_1$, whence the F -basis f_1, f_2 cannot define the given Archimedean order on G .

EXAMPLE 2. Let $G = \langle a_1 \rangle + \langle a_2 \rangle + \langle a_3 \rangle$, where $\langle a_1 \rangle$ is an infinite cyclic group and where $\langle a_2 \rangle + \langle a_3 \rangle$ is isomorphic to the subgroup $H = \langle 1 \rangle + \langle (1/2)(1 + \sqrt{5}) \rangle$ of the additive group of reals. Let $P(G) = \{na_1 \mid n \text{ is a nonnegative integer}\} \cup \{x \mid x \in G - \langle a_1 \rangle, x = ra_1 + sa_2 + ta_3, \text{ and } s + (t/2)(1 + \sqrt{5}) > 0\}$. It readily follows that $P(G)$ defines an order on G , say $\leq$, where for $x, y \in G$, $x \leq y$ if and only if $(-x + y) \in P(G)$. It is easy to see that $0 < a_1$, $0 < a_2$, and $0 < a_3$; also, $P(G)$ is not an Archimedean order on G as $a_1 \ll a_2$ and $a_1 \ll a_3$ (i.e., $na_1 < a_2$, $na_1 < a_3$ for all integers n).

We now define two σ -automorphisms of G :

- (i) $d: a_1 \rightarrow a_1, a_2 \rightarrow a_3, a_3 \rightarrow a_2 + a_3$,
- (ii) $v: a_1 \rightarrow a_1, a_2 \rightarrow a_2 + ma_1, a_3 \rightarrow a_3 + na_1$,

where m, n are arbitrary nonzero integers.

It follows easily that d and v are automorphisms of G . We now show that d is order-preserving: Let $x = ra_1 + sa_2 + ta_3$ and suppose $0 < x$. If $s = t = 0$, then $x^d = ra_1 > 0$. Suppose, therefore, that $0 \neq s$ or $0 \neq t$. Then, without loss of generality, we may assume $r = 0$, so that $x = sa_2 + ta_3$. Then $0 < x$ is equivalent to $0 < s + (t/2)(1 + \sqrt{5})$, which is equivalent to $(-2s)/(1 + \sqrt{5}) < t$. On the other hand, $0 < (sa_2 + ta_3)^d = ta_2 + (s + t)a_3$ is equivalent to $3t + t\sqrt{5} > -s(1 + \sqrt{5})$, which is true if and only if $t > -s(1 + \sqrt{5})/(3 + \sqrt{5}) = -2s/(1 + \sqrt{5})$. Thus, d is order-preserving. An analogous argument establishes that v is also order-preserving, whence d and v are σ -automorphisms of G .

Let $\Delta = \langle d, v \rangle$, so that Δ is a subgroup of the group of σ -automorphisms of G . It follows easily that $v = [v, d]^d$, whence $v \neq 1$ belongs to each term of the lower central series of Δ . Therefore, Δ is not nilpotent, and, as Δ is a subgroup of the group of σ -automorphisms of G , the group of σ -automorphisms of the torsion-free, finitely generated, abelian group G is not nilpotent.

We proceed now with the

PROOF OF THE THEOREM. Let $\{1\} < C_1 < C_2 \cdots < C_n = G$ be the chain of convex subgroups of G with respect to the given order on G .

We observe here that as G satisfies the maximal condition for subgroups, this chain is necessarily of finite length and that this chain is an invariant series of G (see [3]). Let $\theta \in \Delta$. Then θ induces an σ -automorphism on the ordered group C_i/C_{i-1} , for $i=1, 2, \dots, n$, given by $(cC_{i-1})^{\theta'} = c^{\theta}C_{i-1}$.

For each i such that $1 \leq i \leq n$, let Δ_i denote the group of all σ -automorphisms $\theta \in \Delta$ such that θ' centralizes C_i/C_{i-1} , i.e., such that $(cC_{i-1})^{\theta'} = c^{\theta}C_{i-1} = cC_{i-1}$. Then Δ_i is a normal subgroup of Δ for each i , and Δ/Δ_i is isomorphic to a subgroup of the group of σ -automorphisms of C_i/C_{i-1} . But, each C_i/C_{i-1} is an Archimedean ordered group (see [1, p. 50]), whence the group of σ -automorphisms of C_i/C_{i-1} is isomorphic to a subgroup of the multiplicative group of positive real numbers (see [1, Corollary 3, p. 47]). Thus, Δ/Δ_i is abelian for $i=1, 2, \dots, n$. Hence, Δ/Δ_0 is abelian, where $\Delta_0 = \bigcap_{i=1}^n \Delta_i$. Note that Δ_0 centralizes C_i/C_{i-1} for each i , so that $[C_i, \Delta_0] \subseteq C_{i-1}$, $i=1, 2, \dots, n$. Thus, by a result of P. Hall (see [2, Corollary to Theorem 3.8, p. 10]), Δ_0 is nilpotent. Therefore, Δ is nilpotent by abelian.

Smirnov [7] has proved that, for a polycyclic group H , every abelian subgroup of $\text{Aut}(H)$ is finitely generated, whereas Mal'cev [4] has proved that any solvable group, all of whose abelian subgroups are finitely generated, is polycyclic. These results prove that Δ is polycyclic.

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