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A UNIFORMIZATION THEOREM FOR ARBITRARY RIEMANN SURFACES WITH SIGNATURE

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ABSTRACT. An arbitrary Riemann surface with signature can be represented as the quotient of a simply connected Riemann surface by a properly discontinuous group of conformal mappings with the natural projection satisfying some conditions. The representation is unique up to conjugation.

I. **Introduction.** An arbitrary Riemann surface with signature can be represented as the quotient of a simply connected Riemann surface by a properly discontinuous group of conformal mappings with the natural projection satisfying some conditions. The representation is unique up to conjugation. The general idea of part of this result was already mentioned in Koebe [**5**]. But to the knowledge of this author, an accurate statement and a complete proof seem to be still lacking in the literature. The present paper is trying to bridge this gap. The application of this theorem is manifold: special cases can be found in Bers [**1**], Wong [**6**], [**7**].

II. **The Theorem.**

DEFINITION. Let S be a Riemann surface and $\{P_k\}$, $k=1, 2, 3, \cdots$, be a discrete (finite or infinite) sequence of points on S. Let there be an "integer" $\nu_k \geqq 2$ associated with each point P_k (here ν_k may be an actual integer, or the symbol ∞). The following two cases are excluded:

(i) $S=\hat{\mathbf{C}}=\mathbf{C}\cup\{\infty\}$, the Riemann sphere, with one point P and the associated "integer" $\nu\neq\infty$ and

(ii) $S=\hat{\mathbf{C}}$, with two points P_1, P_2 and $\nu_1\neq\nu_2$.

The triple $(S, \{P_k\}, \{\nu_k\})$ is called a Riemann surface with signature. For simplicity, we shall write $\boldsymbol{P}$, $\boldsymbol{\nu}$, $(S, \boldsymbol{P}, \boldsymbol{\nu})$ for $\{P_k\}$, $\{\nu_k\}$, $(S, \{P_k\}, \{\nu_k\})$ respectively.

THEOREM. *Let $(S, \boldsymbol{P}, \boldsymbol{\nu})$ be a Riemann surface with signature. Let $S^\# = S - \bigcup_{\nu_k=\infty}\{P_k\}$, $S^b = S - \bigcup_{\nu_k\geqq 2}\{P_k\}$. Then there exists a simply connected Riemann surface $V^\#$, a properly discontinuous group G of conformal self-mappings of $V^\#$, and a conformal mapping from $V^\#/G$ onto $S^\#$ such that*

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(i) *under this mapping* S^b *is conformally equivalent to* V^b/G, *where* V^b *is* $V^\#$ *with all fixed points of elements of* G *removed, and*

(ii) *the natural projection* $V^\# \to V^\#/G$ *followed by the conformal mapping* $V^\#/G \to S^\#$ *is locally* 1 *to* 1 *at each point of* V^b *and is* ν_k *to* 1 *at the preimages of* P_k *with* $\nu_k < \infty$.

The group G *is determined uniquely up to conjugation in the full group of conformal self-mappings of* $V^\#$.

PROOF. (a) Let $\pi = \pi_1(S^b, b)$ be the fundamental group of S^b with base point b. Let α_k be a closed curve on S^b beginning at b, extending to a point P close to a puncture P_k with $\nu_k < \infty$, then turning around P_k along a small loop d, ν_k times, and returning to P and then to b along the initial path. By saying that the loop is "small" we imply that it encloses no points P_j, $j \neq k$, and if we fill in the puncture P_k, α_k will be homotopic to the identity. Do this for all punctures P_k with $\nu_k < \infty$.

We have a subset $M = \{[\alpha_k]\}$, $\nu_k < \infty$, of π, where $[\alpha_k]$ is the homotopy class of α_k. Let Γ be the smallest normal subgroup of π containing M. Then Γ defines an unbounded, unramified, regular covering (V^b, g) of S^b. (We refer, for instance, to Bers [2].) Essentially, the construction is as follows. Let α, β be curves from b to q on S^b. If $[\alpha\beta^{-1}] \in \Gamma$, we say that α is equivalent to β. Hence all curves beginning at b are divided into equivalence classes $[(\alpha, q)]$. Let V^b be the space whose points are the equivalence class $[(\alpha, q)]$. Define a mapping $g: V^b \to S^b$ by $g([(\alpha, q)]) = q$. One can show that V^b has a natural structure of a Riemann surface and g is a covering projection.

Furthermore, the fundamental group $\pi^* = \pi_1(V^b, \bar{b})$ of V^b with base point $\bar{b}$, which lies over b, is isomorphic to Γ. A closed curve α on S^b beginning at b is lifted to a closed curve $\tilde{\alpha}$ on V^b if and only if $[\alpha] \in \Gamma$. In particular, the closed curve α_k can be lifted to a closed curve $\tilde{\alpha}_k$ on V^b through $\bar{b}$ in the following form. $\tilde{\alpha}_k$ Starts from $\bar{b}$ and extends to a point $\tilde{P}$ close to a puncture $\tilde{P}_k$, where $\tilde{P}$ lies over P and $\tilde{P}_k$ lies over P_k in the sense that a deleted neighborhood of $\tilde{P}_k$ lies over a deleted neighborhood of P_k. (We shall call $\tilde{P}_k$ a "preimage" of P_k under g and we shall justify this terminology later.) The curve then turns around $\tilde{P}_k$ along a small loop $\tilde{d}$ once, where $\tilde{d}$ lies over d, and returns to $\tilde{P}$ and to $\bar{b}$ along the entering curve. Also $\tilde{d}$ encloses no other "preimages", and when we fill in $\tilde{P}_k$, $\tilde{\alpha}_k$ is homotopic to the identity. Also, g is ν_k to 1 in a deleted neighborhood of $\tilde{P}_k$.

Recall that the covering group G of g is the group of conformal self-mappings σ of V^b such that $g \circ \sigma = g$. In the present case, G is isomorphic to π/Γ since Γ is a normal subgroup of π. Also note that G has no fixed points on V^b and G is properly discontinuous.

Since Γ is normal, G is transitive. That is, any two points of V^b over the same point on S^b can be mapped onto each other by an element of G. Therefore we can form the Riemann surface V^b/G and it is conformally equivalent to S^b.

(b) Let $V^\# = V^b \cup \{\text{all "preimages" of } P_k \text{ with } \nu_k < \infty \text{ under } g\}$. We shall show that $V^\#$ is simply connected. If H is the smallest subgroup of π containing M, then Γ is exactly the subgroup generated by $\{xHx^{-1}\}$, $x \in \pi$. Hence each element of Γ can be expressed as a finite product of elements of the form xhx^{-1}, where $h \in H$, $x \in \pi$. Let $\hat{\alpha}$ be a closed curve on $V^\#$ beginning at $\bar{b}$. Then $\hat{\alpha}$ is homotopic to a closed curve $\tilde{\alpha}$ on V^b beginning at $\bar{b}$. When $\tilde{\alpha}$ is projected down by g, we get a closed curve α on S^b such that $[\alpha] \in \Gamma$. Hence $[\alpha]$ can be expressed as a finite product of elements of the form xhx^{-1}, $h \in H$, $x \in \pi$. But h can be expressed as a finite product of elements of M. Therefore the curve α is homotopic to a curve of the form $\beta = \beta_1\beta_2 \cdots \beta_n$, where $\beta_i = c_i\alpha_{k_i}^{m_i} \cdots \alpha_{k_{j_i}}^{m_{j_i}} c_i^{-1}$, $m_i, \cdots, m_{j_i}$, are integers, c_i is some closed curve through b. Since $[\beta] = [\alpha] \in \Gamma$, we can lift β to a closed curve $\tilde{\beta}$ on V^b beginning at $\bar{b}$. And $\tilde{\alpha}$ is homotopic to $\tilde{\beta}$. We shall show that $\tilde{\beta}$ is homotopic to the identity on the filled-in surface $V^\#$, so that $\hat{\alpha}$ is homotopic to the identity on $V^\#$. Since $\hat{\alpha}$ is arbitrary, this will prove that $V^\#$ is simply connected.

First let us assume that $\beta = c_k\alpha_k c_k^{-1}$. V^b is unbounded, hence any closed curve on S^b can be lifted to a closed or open curve on V^b. Lift c_k, α_k, c_k^{-1} to curves $\tilde{c}_k$, $\tilde{\alpha}_k$, $\tilde{d}_k$ on V^b. $\tilde{\alpha}_k$ is a closed curve through $\bar{b}$ such that when we fill in $\tilde{P}_k$, $\tilde{\alpha}_k$ is homotopic to the identity on $V^\#$. $\tilde{c}_k$ is a curve from a point $\bar{b}_1$ to $\bar{b}$, where both $\bar{b}_1$ and $\bar{b}$ lie over the same point b on S^b. $\tilde{d}_k$ is a curve from $\bar{b}$ to $\bar{b}_1$. Since $\tilde{\beta} = \tilde{c}_k\tilde{\alpha}_k\tilde{d}_k$, it follows that $\tilde{\beta}$ is homotopic to $\tilde{c}_k\tilde{d}_k$. But $\tilde{c}_k\tilde{d}_k$ can also be regarded as a curve obtained by lifting the curve $c_kc_k^{-1}$ on S^b. Hence $\tilde{c}_k\tilde{d}_k$ is homotopic to the identity since lifting preserves homotopy. Therefore $\tilde{\beta}$ is homotopic to the identity on $V^\#$.

Secondly, assume that $\beta = c_k\alpha_{k_1}^{m_1} \cdots \alpha_{k_j}^{m_j} c_k^{-1}$. Then β can be expressed as a finite product of elements of the form $c_k\alpha_k c_k^{-1}$. By the previous discussion and the fact that lifting preserves homotopy, we conclude that $\tilde{\beta}$ is homotopic to the identity on $V^\#$ also.

Finally, we consider the general case where $\beta = \beta_1\beta_2 \cdots \beta_n$. Here β_i is as defined before, and the discussion of the previous paragraph can be applied. It follows that $\tilde{\beta}$ is homotopic to the identity on $V^\#$.

(c) Next we show that G is in fact a group of conformal self-mappings from $V^\#$ onto itself. If $\sigma \in G$, then σ maps V^b conformally onto V^b. Consider a deleted neighborhood of a "preimage" $\tilde{P}_k$ of P_k, $\nu_k < \infty$. σ maps this deleted neighborhood conformally onto a deleted neighborhood of another "preimage" $\hat{P}_k$ of P_k. We can certainly extend σ analytically to the puncture $\tilde{P}_k$, sending $\tilde{P}_k$ onto $\hat{P}_k$. In this manner, σ can be extended analytically to all "preimages" so that it is now defined and analytic on $V^\#$. That σ is 1 to 1 on $V^\#$ follows from the fact that σ is 1 to 1 on V^b. And that σ is onto V^b implies that σ is onto $V^\#$. Therefore G is a group of conformal self-mappings of $V^\#$. Also $S^\#$ is conformally equivalent to $V^\#/G$. We can now extend the projection $g: V^b \to S^b$ to $V^\# \to S^\#$ by defining $\tilde{P}_k \mapsto P_k$, where $\tilde{P}_k$ is a "preimage" of P_k with $\nu_k < \infty$. We call this extended projection g also. Then $\tilde{P}_k$ is actually a preimage of P_k under g.

(d) We shall show that a preimage $\tilde{P}_k$ of P_k with $\nu_k < \infty$ is a fixed point of an element $\gamma \in G$ such that $\gamma^{\nu_k} = \mathrm{id}$. Consider V^b and S^b again, so that $\tilde{P}_k$, P_k are punctures. Let $g_{\tilde{D}}$ be the restriction of g to a puncture disk $\tilde{D}$ of $\tilde{P}_k$, so that $g_{\tilde{D}}$ maps $\tilde{D}$ onto a punctured disk D of P_k. In terms of local parameters, we may assume that $\tilde{D}$ is given by $0 < |z| < 1$, D by $0 < |t| < 1$, $t = g_{\tilde{D}}(z)$. $(\tilde{D}, g_{\tilde{D}})$ is in fact an unbounded, unramified, regular covering of D. Let α be a closed curve on D beginning at a point p, turning around P_k exactly once.

On the one hand, we can regard α as a curve on S^b, so that we can lift α to a curve $\tilde{\alpha}$ on $\tilde{D}$ from $\tilde{p}$ to $\tilde{q}$, where both $\tilde{p}$ and $\tilde{q}$ lie over p. Since V^b is a regular covering of S^b, there exists a unique element $\gamma \in G$ such that $\gamma(\tilde{p}) = \tilde{q}$. By the construction of V^b, $\gamma^{\nu_k} = \mathrm{id}$. We shall show that $\tilde{P}_k$ is a fixed point of γ. On the other hand, $\tilde{\alpha}$ can also be regarded as obtained by lifting α in D with respect to $g_{\tilde{D}}$. It follows that there exists a unique element $\gamma_1 \in \tilde{G}$, the covering group for the covering surface $(\tilde{D}, g_{\tilde{D}})$ of D, such that $\gamma_1(\tilde{p}) = \tilde{q}$. By construction, γ_1 is exactly the restriction of γ to $\tilde{D}$. Therefore it suffices to show that γ_1 has $\tilde{P}_k$ as its fixed point.

Let $\pi' = \pi_1(D, p)$ be the fundamental group of D with base point p. It is well known that $\pi' = \{[\alpha^n],\ n = 0, \pm 1, \pm 2, \cdots\}$, where $[\alpha^n]$ is the homotopy class of α^n as usual. Let Γ' be the subgroup of π' generated by $[\alpha']$, where $\alpha' = \alpha^{\nu_k}$. Clearly Γ' is a normal subgroup of π'. Therefore Γ' defines an unbounded, unramified, regular covering $(\hat{D}, \hat{g})$ of D. It is well known that $\hat{D}$ is also a punctured disk, say, $0 < |\zeta| < 1$, $\hat{g}$ is given by $t = \zeta^{\nu_k}$, and the covering group $\hat{G}$ for $(\hat{D}, \hat{g})$ is the finite cyclic group generated by $\hat{\sigma}: \zeta \mapsto e^{2\pi i/\nu_k}\zeta$. Therefore the puncture $\hat{P}_k$ of $\hat{D}$ (i.e., $\zeta = 0$) is a fixed point of $\hat{\sigma}$, and hence a fixed point of all elements of $\hat{G}$.

We now have two coverings $(\tilde{D}, g_{\tilde{D}})$ and $(\hat{D}, \hat{g})$ of D. Let $\tilde{\pi} = \pi_1(\tilde{D}, \tilde{p})$ be the fundamental group of $\tilde{D}$ with base point $\tilde{p}$. The defining subgroup of $(\tilde{D}, g_{\tilde{D}})$ is therefore $g^*_{\tilde{D}}(\tilde{\pi})$, where $g_{\tilde{D}}$ is the induced mapping: $\tilde{\pi} \to \pi'$. Since any closed curve $\tilde{\alpha}$ through $\tilde{p}$ in $\tilde{D}$ is projected onto a closed curve of the form $(\alpha')^m$, m an integer, $g_{\tilde D}^*(\tilde\pi)\supset\Gamma'$. It follows that $(\tilde D, g_{\tilde D})$ is a covering of D subordinate to $(\hat D, \hat g)$.

We have the following commutative diagram:

$$\begin{tikzcd} \tilde{D} \arrow[d, "g_{\tilde D}"'] & \hat{D} \arrow[l, "\tau"'] \arrow[dl, "\hat g"] \\ D & \end{tikzcd}$$

where τ is a covering projection from $\hat D$ onto $\tilde D$, and $(\hat D, \tau)$ is an unbounded, unramified covering of $\tilde D$. We can extend τ analytically to $\hat P_k$, $\hat g$ to $\hat P_k$ and $g_{\tilde D}$ to $\tilde P_k$ such that $\tau(\hat P_k)=\tilde P_k$, $\hat g(\hat P_k)=P_k$, and $g_{\tilde D}(\tilde P_k)=P_k$. Therefore we have $g_{\tilde D}\circ\tau=\hat g$ on $\hat D\cup\{\hat P_k\}$. τ is now a holomorphic function from the disk $\hat D\cup\{\hat P_k\}$ onto the disk $\tilde D\cup\{\tilde P_k\}$. By the monodromy theorem, τ must be 1 to 1, hence conformal. It follows that $\tilde G=\tau\circ\hat G\circ\tau^{-1}$. Since the elements of $\hat G$ have fixed point $\hat P_k$, the elements of $\tilde G$ have fixed point $\tilde P_k$. In particular, γ_1 has fixed point $\tilde P_k$.

(e) Finally, we shall show the "uniqueness" of G. That is, if G_1 is any properly discontinuous group of conformal self-mappings of $V^\#$ such that

(i) $V^\#/G_1$ is conformally equivalent to $S^\#$ (under this conformal mapping, S^b is conformally equivalent to U^b/G_1, where U^b is $V^\#$ with all fixed points of elements of G_1 removed), and

(ii) the natural projection $V^\#\to V^\#/G_1$ followed by the conformal mapping $V^\#/G_1\to S^\#$ is locally 1 to 1 at each point of U^b and is ν_k to 1 at the preimages of P_k with $\nu_k<\infty$, then G_1 is conjugate to G in the full group of conformal self-mappings of $V^\#$.

Denote by g_1' the composite mapping of the natural projection $V^\#\to V^\#/G_1$ followed by the conformal mapping $V^\#/G_1\to S^\#$. (U^b, g_1) is an unbounded, unramified, regular covering of S^b. Its defining subgroup $\Gamma_1=g_1^*(\pi_1)$ is a normal subgroup of π (recall that $\pi=\pi_1(S^b, b)$), where $\pi_1=\pi_1(U^b, \bar b_1)$ is the fundamental group of U^b with base point $\bar b_1$, $g_1(\bar b_1)=b$, and $g_1^*:\pi_1\to\pi$ is the induced mapping. Also, G_1 is the covering group for (U^b, g_1).

We claim that $\Gamma_1\supset\Gamma$. Let $\hat P_k$ be a point lying over P_k with $\nu_k<\infty$ under g_1. Let β_k be a closed curve on U^b beginning at $\bar b_1$ extending to a point $\hat P$ close to $\hat P_k$, turning around $\hat P_k$ once in a small loop which encloses no other preimages of any points P_k with $\nu_k<\infty$, and then returning through $\hat P$ to $\bar b_1$. $[\beta_k]\in\pi_1$, and the projection of β_k under g_1 is a curve homotopic to α_k. Therefore $g_1^*([\beta_k])=[\alpha_k]$. Do this for all P_k with $\nu_k<\infty$. We conclude that $\Gamma_1\supset M$; hence $\Gamma_1\supset\Gamma$. It follows that

(U^b, g_1) is a covering of S^b subordinate to (V^b, g). We have the following commutative diagram:

$$\begin{tikzcd} U^b \arrow[d, "g_1"'] & V^b \arrow[l, "\delta"'] \arrow[dl, "g"] \\ S^b & \end{tikzcd}$$

where δ is a covering projection, and (V^b, δ) is an unbounded, unramified covering of U^b.

Let $\hat{V}^b = V^b \cup \{$all preimages of P_k with $\nu_k < \infty$ under $g\}$. Let $\hat{U}^b = U^b \cup \{$all preimages of P_k with $\nu_k < \infty$ under $g_1\}$. $\hat{V}^b = \hat{U}^b = V^\#$. We use different notations here just for the sake of easier distinction. Clearly we can extend δ analytically to $\hat{V}^b$ in a natural way, so that $\hat{V}^b$ is mapped onto $\hat{U}^b$ and $g_1 \circ \delta = g$ on $\hat{V}^b$. Since $\hat{U}^b = V^\#$ is simply connected, by monodromy theorem δ must be 1 to 1, hence δ is a conformal self-mapping of $V^\#$. It follows that $G_1 = \delta \circ G \circ \delta^{-1}$. Proof is complete.

REMARK 1. The fact that $V^\#$ is simply connected implies that $V^\#$ is conformally equivalent to $\hat{\mathbf{C}}$, $\mathbf{C}$ or Δ, where Δ denotes the unit disk. We can actually enumerate all Riemann surfaces with signature $(S, \boldsymbol{P}, \boldsymbol{\nu})$ such that $V^\#$ is conformally equivalent to $\hat{\mathbf{C}}$ or $\mathbf{C}$ as follows. (See Ford [3, Chapter IX, §§94, 95].)

(i) $S = \hat{\mathbf{C}}$ with one point P, $\nu = \infty$,

(ii) $S = \hat{\mathbf{C}}$ with two points P_1, P_2, and $\nu_1 = \nu_2$,

(iii) $S = \hat{\mathbf{C}}$ with three points P_1, P_2, P_3, and $1/\nu_1 + 1/\nu_2 + 1/\nu_3 \geqq 1$,

(iv) $S = \hat{\mathbf{C}}$ with four points P_1, P_2, P_3, P_4 and $\nu_1 = \nu_2 = \nu_3 = \nu_4 = 2$,

(v) $S =$ a torus and there are no points P_k.

We shall call these cases exceptional.

REMARK 2. For all nonexceptional cases, $V^\#$ is conformally equivalent to Δ. We therefore have the following result.

COROLLARY. *Given a nonexceptional Riemann surface with signature* $(S, \boldsymbol{P}, \boldsymbol{\nu})$, *there exists a Fuchsian group* G *with the unit circle as its fixed circle such that* $S^\# = S - \bigcup_{\nu_k = \infty} \{P_k\}$ *is conformally equivalent to* Δ/G. $S^b = S - \bigcup_{\nu_k \geqq 2} \{P_k\}$ *is conformally equivalent to* Δ_G/G, *where* $\Delta_G = \Delta - \{$*all elliptic fixed points of* $G\}$, *and* S *is conformally equivalent to* $\hat{\Delta}_G/G$, *where* $\hat{\Delta}_G = \Delta \cup \{$*all parabolic fixed points of* $G\}$. *The natural projection* $\Delta \to \Delta/G$ *followed by the conformal mapping* $\Delta/G \to S^\#$ *is locally* 1 *to* 1 *at each point of* Δ_G, *and is* ν_k *to* 1 *at the preimages of* P_k *with* $\nu_k < \infty$.

The Fuchsian group G *is determined uniquely up to conjugation by a Möbius transformation.*

PROOF. We have only to show that S is conformally equivalent to $\hat{\Delta}_G/G$. But this is exactly a result of Heins. (See Heins [4].)

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