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CHARACTERIZATION OF RINGS USING QUASIPROJECTIVE MODULES. II

JONATHAN S. GOLAN[1]

ABSTRACT. Semiperfect rings, semihereditary rings, and hereditary rings, are characterized by properties of quasiprojective modules over their matrix rings.

In [4], we characterized semisimple artinian, semiperfect, and perfect rings by the behavior of quasiprojective left R-modules over them. In this paper we will continue this method of characterization. As before, R will always denote an associative ring with 1 and all modules and morphisms will be taken from the category of unitary left R-modules unless otherwise specified.

Recall that a module M is *quasiprojective* iff, for every epimorphism $\lambda: M \to N$, $\mathrm{Hom}(M, \lambda): \mathrm{Hom}(M, M) \to \mathrm{Hom}(M, N)$ is also an epimorphism. Basic facts on quasiprojective modules can be found in [6] or [8].

An epimorphism $\mu: U \to M$ is a *projective cover* of M iff U is projective and $\ker(\mu)$ is small in U. (A is small in B iff $A+C=B$ implies $B=C$); it is a *quasiprojective cover* iff (i) U is quasiprojective; (ii) $\ker(\mu)$ is small in U, and (iii) U/V is not quasiprojective for all nonzero submodules V of $\ker(\mu)$. If M has a projective cover then it has a quasiprojective cover unique up to isomorphism [8, Proposition 2.6].

We will also need the following facts about quasiprojective modules: If M is quasiprojective then so is M^n (the direct sum of n copies of M) [7]. If M is quasiprojective and N is a stable submodule of M (that is to say, $N\alpha \subseteq N$ for any endomorphism α of M), then M/N is also quasiprojective.

1. **A change-of-rings theorem.** Let R, S be associative rings with 1 and let $T: R\text{-mod} \to S\text{-mod}$ be a covariant functor from the category of all unitary left R-modules to the category of all unitary left S-modules. Let $\mathfrak{M}$ be a full subcategory of R-mod. Then T is called a *local category equivalence at* $\mathfrak{M}$ iff there exists a covariant functor

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T': S-mod$\to R$-mod such that the pair consisting of the restriction of T to $\mathfrak{M}$ and the restriction of T' to $T(\mathfrak{M})$ is a category equivalence. That is to say, iff $T'T$ and TT' are naturally equivalent to the respective identity functors on $\mathfrak{M}$ and $T(\mathfrak{M})$.

1.1 THEOREM. *Let M be a left R-module and $\mathfrak{M}$ the full subcategory of R-mod the objects of which are all homomorphic images of M. Let T: R-mod$\to S$-mod be a local category equivalence at $\mathfrak{M}$. Then M is quasiprojective iff $T(M)$ is quasiprojective.*

PROOF. Assume M is quasiprojective and let $\alpha: M\to N$ be an R-epimorphism. Then for each S-homomorphism $\beta: T(M)\to T(N)$ there exists an R-endomorphism ξ of M making the diagram

$$\begin{tikzcd}
& M \arrow[rr, "\cong"] \arrow[dl, "\xi"'] & & T'T(M) \arrow[d, "T'(\beta)"] \\
M \arrow[r, "\alpha"] & N \arrow[rr, "\cong"] & & T'T(N)
\end{tikzcd}$$

commute. Applying T, we obtain in turn the commutative diagram

$$\begin{tikzcd}
& T(M) \arrow[r, "\cong"] \arrow[dl, "T(\xi)"'] \arrow[d, "\beta"] & TT'T(M) \arrow[d, "TT'(\beta)"] \\
T(M) \arrow[r, "T(\alpha)"] & T(N) \arrow[r, "\cong"] & TT'T(N)
\end{tikzcd}$$

proving that $T(M)$ is quasiprojective. Conversely, if we assume $T(M)$ is quasiprojective then, applying the same argument we show that $T'T(M)\cong M$ is quasiprojective.

We now apply this theorem to two specific cases:

(I) Let R be a ring and $S=R_n$, the full ring of $n\times n$ matrices over R. If $\alpha: M\to N$ is an R-homomorphism, then α induces an S-homomorphism $\alpha': M^n\to N^n$ defined by $(m_1, \cdots, m_n)\alpha'=(m_1\alpha, \cdots, m_n\alpha)$. Conversely, if $e_{11}\in S$ is the matrix the (1, 1)-entry of which equals 1_R and all other entries of which are 0, and if $\beta: U\to V$ is an S-homomorphism, then the restriction of β induces an R-homomorphism $\beta'': e_{11}U\to e_{11}V$. The functors T: R-mod$\to S$-mod and T': S-mod $\to R$-mod given by $T(M)=M^n$ and $T(\alpha)=\alpha'$, $T'(U)=e_{11}U$ and $T'(\beta)=\beta''$ are category equivalences (see [5] for details). We therefore have:

1.2 COROLLARY. *Let R be a ring and $S=R_n$. Then*
(1) *${}_RM$ is quasiprojective iff ${}_S(M^n)$ is quasiprojective.*
(2) *${}_SU$ is quasiprojective iff ${}_R(e_{11}U)$ is quasiprojective.*

(II) Let I be a two-sided ideal of a ring R and let $S=R/I$. Define the functor $T:R$-mod$\rightarrow S$-mod by $T(M)=M/IM$ and, if $\alpha:M\rightarrow N$ is an R-homomorphism, $T(\alpha)=\bar{\alpha}$, where $(m+IM)\bar{\alpha}=m\alpha+IM$. Conversely, every left S-module U can be considered as a left R-module and every S-homomorphism as an R-homomorphism. This gives us a functor $T':S$-mod$\rightarrow R$-mod. TT' is the identity functor on S-mod. On the other hand, if M is a left R-module the annihilator of which contains I, then $T'T(N)=N$ for all epimorphic images N of M. We therefore have:

1.3 COROLLARY. *Let M be a left R-module and I a two-sided ideal of R contained in the annihilator of M. Then M is quasiprojective over R iff it is quasiprojective over R/I.*

2. **The basic tool.** In [4] we proved the following result, the proof of which we shall restate for completeness:

2.1 LEMMA. *A sufficient condition for an epimorphism $\lambda:U\rightarrow M$ to split is that $U\oplus M$ be quasiprojective.*

Proof. Let i_U, i_M [resp. π_U, π_M] be the canonical inclusions into [resp. projections from] $U\oplus M$. Then $\pi_U\lambda:U\oplus M\rightarrow M$ is an epimorphism and so, by quasiprojectivity, there exists an endomorphism ξ of $U\oplus M$ such that $\pi_M=\xi\pi_U\lambda$. Then $(i_M\xi\pi_U)\lambda=i_M\pi_M=$ identity on M, implying that λ splits.

2.2 THEOREM. *Let $\lambda:P\rightarrow M$ be an epimorphism from a projective module P onto a module M. Then*
(1) *M is projective iff $P\oplus M$ is quasiprojective.*
(2) *M has a projective cover iff $P\oplus M$ has a quasiprojective cover.*

PROOF. (1) follows immediately from Lemma 2.1. As for (2), if M has a projective cover $\mu:P'\rightarrow M$ then $\mathrm{id}_P\oplus\mu:P\oplus P'\rightarrow P\oplus M$ is a projective cover and so, as remarked above, $P\oplus M$ has a quasiprojective cover.

Conversely, assume $P\oplus M$ has a quasiprojective cover $\mu:Q\rightarrow P\oplus M$. Then the epimorphism $\mu\pi_P:Q\rightarrow P$ splits by the projectivity of P and so $Q\cong P\oplus W$. Without loss of generality we can therefore assume $Q=P\oplus W$ and $\mu=\mathrm{id}_P\oplus\mu'$, where μ' is the restriction of μ to W. Ker(μ') is a homomorphic image of ker(μ) and so is small in W. Furthermore, $\mu':W\rightarrow M$ is an epimorphism.

By the projectivity of P there exists a homorphism $\beta: P \to W$ such that $\lambda = \beta\mu'$. Since λ is an epimorphism, $W = P\beta + \ker(\mu') = P\beta$ by smallness of $\ker(\mu')$. Since $P \oplus W$ is quasiprojective, β splits by Lemma 2.1 and so W is isomorphic to a direct summand of P and hence is projective. This proves that $\mu': W \to M$ is a projective cover.

Note. The above proof is based on a proof communicated to the author by Anne Koehler.

3. **Semiperfect rings.** A ring R is [*semi-*] *perfect* iff every [cyclic] left R-module has a projective cover. In [**4**] we characterized [semi-] perfect rings as rings over which every [finitely-generated] module has a quasiprojective cover. The class of rings over which every cyclic left R-module has a quasiprojective cover is considerably larger and includes, for example, all commutative rings. (In fact, if R is commutative and I an ideal of R, then I is stable and so R/I is quasiprojective.) However, we do have the following characterization:

(3.1) THEOREM. *The following are equivalent for a ring R:*

(1) *R is semiperfect.*

(2) *For all $n \geqq 1$, every cyclic R_n-module has a quasiprojective cover.*

(3) *There exists an $n > 1$ such that every cyclic R_n-module has a quasiprojective cover.*

PROOF (1)$\Rightarrow$(2) follows from the fact that if R is semiperfect so is R_n for all $n \geqq 1$ [**5**, Theorem 3] and (2)$\Rightarrow$(3) is trivial. Therefore assume (3) and let $n > 1$ satisfy the condition that every cyclic R_n-module has a quasiprojective cover. Let L be a left ideal of R, L_n the left ideal of R_n consisting of all matrices with entries from L. Let $e_{ij} \in R_n$ be the matrix with 1_R in the (i, j) position and zeros elsewhere. Then $R_n/L_n e_{11}$ is isomorphic to $P \oplus M$, where $M = R_n e_{11}/L_n e_{11}$ and $P = \sum_{i=2}^{n} R_n e_{ii}$. P is clearly R_n-projective and the map $\lambda: P \to M$ which sends $[a_{ij}]$ to $[a_{ij}]e_{21} + L_n e_{11}$ is an R_n-epimorphism. Since $P \oplus M$ has a quasiprojective cover, by Theorem 2.2(2), M has a projective cover $\mu: W \to M$ over R_n. $(e_{11}W)\mu = e_{11}(W\mu) = e_{11}M$ which is isomorphic, as an R-module, to R/L. W is R_n-projective and so $e_{11}W$ is R-projective [**5**]. The induced R-homomorphism $\mu': e_{11}W \to R/L$ is then a projective cover, proving (1).

4. **Hereditary and semihereditary rings.** A ring R is *left* [*semi-*] *hereditary* iff every [finitely-generated] left ideal of R is projective. Equivalently, R is left [semi-] hereditary iff every [finitely-generated] submodule of a projective left R-module is projective [**1**, pp. 14–15]. R is a *left PP-ring* iff every principal left ideal of R is projective.

We will need the following result of Colby and Rutter [**2**, Propositions 2.3 and 2.4]:

4.1 THEOREM. *A ring R is left [semi-] hereditary iff the endomorphism ring of every [finitely-generated] free left R-module is a left PP-ring.*

4.2 LEMMA. *A ring is a left PP-ring iff every principal left ideal of R_2 generated by a diagonal matrix is quasiprojective.*

PROOF. Let R be a left PP-ring and let K be the left ideal of R_2 generated by $\left[\begin{smallmatrix} a & 0 \\ 0 & b \end{smallmatrix}\right]$. Then, by Corollary 1.2, K is quasiprojective over R_2 iff $e_{11}K \cong Ra \oplus Rb$ is quasiprojective over R, which is the case since R is left PP. Conversely, let $a \in R$ and let K be the principal left ideal of R_2 generated by $\left[\begin{smallmatrix} a & 0 \\ 0 & 1 \end{smallmatrix}\right]$. Then K is quasiprojective over R_2 and so $e_{11}K \cong Ra \oplus R$ is quasiprojective over R. Since R maps epimorphically onto Ra, this implies that Ra is projective by Theorem 2.2.

4.3 THEOREM. *The following are equivalent for a ring R:*

(1) *R is left semihereditary.*

(2) *Every finitely-generated submodule of a projective left R-module is quasiprojective.*

(3) *Every finitely-generated left ideal of R_n is quasiprojective, for all $n \geqq 1$.*

(4) *Every principal left ideal of R_n is quasiprojective, for all $n \geqq 1$.*

PROOF. (1)$\Rightarrow$(2) and (3)$\Rightarrow$(4) are trivial. (2)$\Rightarrow$(1): Assume (2) and let N be a finitely-generated submodule of a projective left R-module P. Then there exists a finitely-generated projective module P' which maps epimorphically onto N. $P' \oplus N$ is then a finitely-generated submodule of the projective module $P' \oplus P$ and so is quasiprojective. By Proposition 2.2 this implies that N is projective, proving (1).

(1)$\Rightarrow$(3) follows since, if R is left semihereditary, so is R_n for all $n \geqq 1$ [**5**]. (4)$\Rightarrow$(1): By Lemma 4.2, (4) implies that R_n is a left PP-ring for all $n \geqq 1$ and so (1) follows by Theorem 4.1.

4.4 THEOREM. *The following are equivalent for a ring R:*

(1) *R is left hereditary.*

(2) *Every submodule of a projective left R-module is quasiprojective.*

(3) *Every principal left ideal of E is quasiprojective, where E is the endomorphism ring of a free R-module.*

PROOF. The proof is along the same lines as that of Theorem 4.3, remembering that if M is a free module with endomorphism ring E, $M \oplus M$ is free with endomorphism ring isomorphic to E_2.

5. **Rings over which submodules of quasiprojectives are quasiprojective**. By Theorems 4.3 and 4.4., a sufficient condition for R to be left [semi-] hereditary is that every [finitely-generated] submodule of a quasiprojective left R-module be quasiprojective. The converse is not true. To see this, let $\boldsymbol{Z}$ be the ring of integers, which is left hereditary. Then $8\boldsymbol{Z}$ is a stable submodule of $\boldsymbol{Z}$ and so $\boldsymbol{Z}/8\boldsymbol{Z}$ is quasiprojective over $\boldsymbol{Z}$. Hence so is $M=\boldsymbol{Z}/8\boldsymbol{Z}\oplus\boldsymbol{Z}/8\boldsymbol{Z}$. Let $N=2\boldsymbol{Z}/8\boldsymbol{Z}\oplus\boldsymbol{Z}/8\boldsymbol{Z}\subseteq M$. Then the epimorphism $\lambda:\boldsymbol{Z}/8\boldsymbol{Z}\to 2\boldsymbol{Z}/8\boldsymbol{Z}$ ($x\lambda=2x$) does not split and so N is not quasiprojective.

5.1 THEOREM. *Let R be a ring over which [finitely-generated] submodules of quasiprojective modules are quasiprojective. Then every factor ring of R is left [semi-] hereditary. If R is left perfect then the converse also holds.*

PROOF. Let I be a two-sided ideal of R, $S=R/I$. Let P be a projective left S-module with [finitely-generated] submodule M. By Corollary 1.3, P is quasiprojective as a left R-module and hence, by hypothesis, so is M. M is then quasiprojective as a left S-module. By Theorems 4.3 and 4.4, this proves that S is left [semi-] hereditary.

Conversely, assume that R is left perfect and let Q be a quasiprojective left R-module with [finitely-generated] submodule M. Let I be the annihilator of Q in R, $S=R/I$. Since R is left perfect, Q has a projective cover and so is projective over S [**3**, Theorem 2.3]. By assumption S is left [semi-] hereditary and so M is projective over S. By Corollary 1.3, M is then quasiprojective over R.

5.2 THEOREM. *The class of rings over which [finitely-generated] submodule of quasiprojective modules are quasiprojective is closed under taking factor rings and matrix rings.*

PROOF. By an easy application of Corollaries 1.2 and 1.3.

ADDED IN PROOF. It has been called to the author's attention that the results credited to [**2**] were first proven by Stephenson and Tsukerman, *Endomorphism rings of projective modules*, Siberian Math. J. **11** (1970), 228–232. (Russian)

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THE HEBREW UNIVERSITY OF JERUSALEM, JERUSALEM, ISRAEL