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A DISTORTION THEOREM FOR ANALYTIC FUNCTIONS

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ABSTRACT. Let $f(z)$ be a function analytic in the disk $E\{z: |z| <1\}$ and for some real number $n>0$ let $|f(z)| \leqq (1-|z|^2)^{-n}$, $z\in E$. In this paper it is shown that

$$|f'(z)| \leqq \frac{(n+1)^{n+1}}{n^n}\left[1-\left(\frac{n}{n+1}\right)^{2n}(1-|z|^2)^{2n}|f(z)|^2\right] \div (1-|z|^2)^{n+1},$$

$z\in E$. In the special case $n=1$ there is a constant K, $3\leqq K\leqq 4$, so that

$$|f'(z)| + |f(z)|^2 \leqq K(1-|z|^2)^{-2}.$$

This result has application in univalent function theory.

1. **Introduction.** For functions $f(z)$, analytic and bounded in modulus by one on the disk $E\{z: |z| <1\}$ it is well known that $|f'(z)|$, the modulus of the derived function, satisfies the inequality

$$|f'(z)| \leqq \frac{1-|f(z)|^2}{1-|z|^2}, \qquad z\in E. \tag{1.1}$$

Recently Duren, Shapiro and Shields [2] have sketched a simple proof, using a contour integral representation of $f(z)$, that whenever $f(z)$ satisfies the growth inequality

$$|f(z)| \leqq (1-|z|^2)^{-1}, \qquad z\in E, \tag{1.2}$$

then

$$|f'(z)| \leqq C(1-|z|^2)^{-2}, \qquad z\in E, \tag{1.3}$$

with the constant $C\leqq 4$. The estimate (1.3) is very useful in connection with computations involving the Schwarzian derivative of an analytic function and problems relating to the univalency of such functions (see for example [1], [2], [3], [4], [5]). The best or smallest value of the constant C is apparently still unknown [see Research Problem N, Bull. Amer. Math. Soc. **71** (1965), 857].

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In this note we extend the result given in (1.3) in several ways and include both (1.1) and (1.3) as special cases of a more general formulation. The method of proof appears to be new. It starts with the inequality (1.1) and avoids the integral representation method of the authors in [2] to obtain (1.3). In this way an additional term improves (1.3) (see Corollary 3). The basic result appears in Theorem A.

THEOREM A. *Let $f(z)$ be a function analytic in the disk $E\{z: |z| < 1\}$ and for some given real number $n > 0$ let*

$$|f(z)| \leqq (1 - |z|^2)^{-n}, \qquad z \in E.$$

Let $A(r)$, $B(r)$, $C(r)$ be the nonnegative, continuous real functions of r defined for $0 \leqq r \leqq 1$ by the equations

$$\begin{aligned} A(r) &= [(2n-1)^2 r^4 + (12n+2) r^2 + 1]^{1/2}, \\ B(r) &= [4n + 1 - (2n-1) r^2 - A(r)]/(4n+2), \\ C(r) &= [4n + 1 - (2n-1) r^2 + A(r)]^n \cdot [(2n+3) r^2 - 1 + A(r)] \\ &\quad \cdot [(2n-1) r^2 + A(r) + 1]^{1/2} \div (4n)^n (4r^2)(4n+2)^{1/2}, \\ C(0) &= \lim_{r \to 0} C(r) = (1 + 1/2n)^n \cdot (2n+1)^{1/2}. \end{aligned}$$

Then for $z \in E$

$$\begin{aligned} |f'(z)| &\leqq \frac{C(|z|)[1 - B(|z|)^{2n} |f(z)|^2]}{(1 - |z|^2)^{n+1}} \\ &\leqq \frac{C(|z|)[1 - (n/(n+1))^{2n} \cdot (1 - |z|^2)^{2n} \cdot |f(z)|^2]}{(1 - |z|^2)^{n+1}}. \end{aligned}$$

Moreover,

$$C(|z|) \leqq C(1) = (n+1)^{n+1}/n^n < e(n+1), \qquad 0 \leqq |z| \leqq 1.$$

COROLLARY 1. *The inequality* (1.1) *follows from Theorem* A *as a limiting case as $n \to 0$ whenever $f(z)$ is analytic with bounded modulus, $|f(z)| \leqq 1$, $z \in E$.*

COROLLARY 2. *If $f(z)$ is analytic in E and if, for some $n > 0$, $|f(z)| \leqq (1 - |z|^2)^{-n}$, $z \in E$, then there exists a smallest constant A, independent of $f(z)$ and n, $1 \leqq A \leqq e$, such that*

$$|f'(z)| \leqq (n+1) A (1 - |z|^2)^{-n-1}, \qquad z \in E.$$

COROLLARY 3. *If $f(z)$ is analytic and if $|f(z)| \leqq (1 - |z|^2)^{-1}$, $z \in E$, then there exists a smallest absolute constant K, $3 \leqq K \leqq 4$, such that*

$$|f'(z)| + |f(z)|^2 \leqq K(1 - |z|^2)^{-2}.$$

2. **Proofs.** Let n be an arbitrary positive number. Let $f(z)$ be analytic in E and satisfy the inequality

$$|f(z)| \leqq (1 - |z|^2)^{-n}, \qquad z \in E. \tag{2.1}$$

Let ρ be a real number in the open interval $(0, 1)$ and define the analytic function $\phi(z)$ by the equation $\phi(z) = (1-\rho^2)^n \cdot f(\rho z)$. Then $\phi(z)$ is analytic on the closure of E and $|\phi(z)| \leqq 1$ for $|z| \leqq 1$. It follows from the well-known inequality for bounded analytic functions:

$$|\phi'(z)| \leqq \frac{1 - |\phi(z)|^2}{1 - |z|^2}, \qquad z \in E, \tag{2.2}$$

that

$$\rho(1 - \rho^2)^n |f'(\rho z)| \leqq \frac{1 - (1 - \rho^2)^{2n} \cdot |f(\rho z)|^2}{1 - |z|^2}, \qquad |z| < 1.$$

Let $z = \rho^k e^{(k+1)i\phi}$, $(k+1)\phi = \theta$, $\rho^{k+1} = r$ where $k > 0$. Then $\rho z = re^{i\theta}$ and

$$|f'(re^{i\theta})| \leqq \frac{1 - (1 - r^{2/(k+1)})^{2n} \cdot |f(re^{i\theta})|^2}{r^{1/(k+1)} \cdot (1 - r^{2/(k+1)})^n \cdot (1 - r^{2k/(k+1)})} \tag{2.3}$$

for $0 < r < 1$ and $0 \leqq \theta \leqq 2\pi$. For fixed r we now choose $k > 0$ so that the denominator

$$y = (r^{1/(k+1)} - r^{(2k+1)/(k+1)})(1 - r^{2/(k+1)})^n$$

is maximized. For $0 < r < 1$ and $r^2 = x^{k+1}$, $0 < x < 1$, $k > 0$, $dy/dx = -P(k, r) \cdot Q(x)$ where

$$P(k, r) = \frac{1}{(k+1)^2}\left(\log \frac{1}{r}\right) x^{-1/2}(1 - x)^{n-1} > 0$$

and

$$Q(x) = (2n + 1)x^2 - (1 + (2n - 1)r^2)x - r^2.$$

Then when $k = 0$, we have $x = r^2$ and $Q(r^2) = -2r^2(1 - r^2) < 0$, $dy/dx > 0$. When $k = +\infty$, we have $x = 1$ and $Q(1) = 2n(1 - r^2) > 0$. Thus the maximum of y is attained by choosing x the sole positive root of $Q(x) = 0$.

$$r^{2/(k+1)} = x = \frac{1 + (2n - 1)r^2 + [(2n - 1)^2 r^4 + (12n + 2)r^2 + 1]^{1/2}}{4n + 2}. \tag{2.4}$$

This means that y is maximized by the following choice of k:

$$k = \left[\frac{2\log(1/r)}{\log\{(A(r) - 1 - (2n-1)r^2)/2r^2\}} - 1\right] \tag{2.5}$$

where

$$A(r) = [(2n-1)^2r^4 + (12n+2)r^2 + 1]^{1/2}. \tag{2.6}$$

It is clear by the method of construction that $k>0$ and hence $r\in(0, 1)$ implies that $\rho = r^{1/(k+1)}\in(0, 1)$.

For the choice of k given in (2.5) we have

$$1 - r^{2/(k+1)} = B(r) = \frac{4n+1-(2n-1)r^2 - A(r)}{4n+2}$$

$$= \frac{4n(1-r^2)}{4n+1-(2n-1)r^2+A(r)},$$

$$(1 - r^{2/(k+1)})^{-n} = \left(\frac{4n+1-(2n-1)r^2+A(r)}{4n(1-r^2)}\right)^n,$$

$$(1 - r^{2k/(k+1)})^{-1} = \frac{[1+(2n-1)r^2+A(r)][(2n+3)r^2-1+A(r)]}{8(2n+1)r^2(1-r^2)}.$$

For the choice of k given in (2.5) we now have from (2.3) the inequality

$$\begin{aligned} |f'(re^{i\theta})| &\leqq \frac{1-(1-r^{2/(k+1)})^{2n}\cdot|f(re^{i\theta})|^2}{r^{1/(k+1)}\cdot(1-r^{2/(k+1)})^n\cdot(1-r^{2k/(k+1)})} \\ &= \frac{C(r)}{(1-r^2)^{n+1}}[1-(B(r))^{2n}\cdot|f(re^{i\theta})|^2], \end{aligned} \tag{2.7}$$

where $A(r)$, $B(r)$ and $C(r)$ are defined as in the statement of Theorem A. It is easily seen that $C(1) = (n+1)^{n+1}/n^n$, $B(1) = 0$, $A(1) = 2n+2$. We shall show that $B(r)\geqq(n/(n+1))(1-r^2)$ and $C(r)\leqq C(1)$ for $n>0$, $0\leqq r\leqq 1$.

The inequality $B(r)\geqq(n/(n+1))(1-r^2)$, or

$$4n+1-(2n-1)r^2 - [(2n-1)^2r^4+(12n+2)r^2+1]^{1/2} \geqq (4n+2)(n/(n+1))(1-r^2),$$

reduces after simplification and squaring to the simple inequality $(8n^2+4n)(1-r^2)^2\geqq 0$ which is satisfied for $n>0$ and $0\leqq r\leqq 1$.

The second inequality $C(r)\leqq C(1) = (n+1)^{n+1}/n^n$ for $n>0$ is equivalent to the inequality

$$[4n + 1 - (2n - 1)r^2 + A(r)]^n \cdot [(2n + 3)r^2 - 1 + A(r)] \cdot [(2n - 1)r^2 + 1 + A(r)]^{1/2} \leqq (4n + 4)^{n+1} \cdot (4n + 2)^{1/2} \cdot r^2$$

where $A(r)$ is given by (2.6). Since $n > 0$ it is easily verified that

$$0 < 4n + 1 - (2n - 1)r^2 + A(r) \leqq (4n + 4)$$

with equality only for $r = 1$. Hence the factor

$$[4n + 1 - (2n - 1)r^2 + A(r)]^n \leqq (4n + 4)^n, \qquad n > 0.$$

Therefore, in order to obtain $C(r) \leqq C(1)$ it is sufficient to show that

$$[(2n + 3)r^2 - 1 + A(r)][(2n - 1)r^2 + 1 + A(r)]^{1/2} \leqq (4n + 4)(4n + 2)^{1/2} \cdot r^2$$

for $n > 0$ and $0 \leqq r \leqq 1$. Squaring and simplifying we obtain

$$[1 + (2n + 1)r^2](2n + 1)A(r) \leqq (2n + 1)[1 + 2(2n + 1)^2 r^2 - (4n^2 - 1)r^4].$$

Since the right-hand side of this last inequality is positive for $0 \leqq r < 1$ we may square again and obtain finally the inequality

$$16n^2(1 - r^2)[1 + (4n^2 + 8n + 3)r^2] \geqq 0, \qquad n > 0, \quad 0 \leqq r \leqq 1.$$

This completes the proof that $C(r) \leqq C(1) = n^{-n}(n+1)^{n+1}$ for $n > 0$, and that Theorem A holds.

From Theorem A we have in particular, that if

$$|f(z)| \leqq (1 - |z|^2)^{-n}, \qquad n > 0, \quad |z| < 1, \tag{2.8}$$

then

$$|f'(z)| \leqq \left(\frac{n + 1}{n}\right)^n \cdot \frac{n + 1}{(1 - |z|^2)^{n+1}} < e(n + 1)(1 - |z|^2)^{-n-1}. \tag{2.9}$$

The factor e cannot be replaced by a constant A less than one as the following example shows. Let $n > 0$ be chosen arbitrarily small. Let $0 < x < 1$. Then let

$$f(z) = \left(\frac{x + z}{1 + x^{z}}\right)(1 - z^2)^{-n}, \qquad |z| < 1. \tag{2.10}$$

It follows that $|f(z)| \leqq (1 - |z|^2)^{-n}$, $|z| < 1$. Moreover $f'(-x) = (1 - x^2)^{-n-1}$.

If the factor e could be replaced by a constant $A<1$ the given example shows that we would then have $1<A(n+1)$ for arbitrarily small $n>0$, a contradiction.

For large values of n the estimate $A(n+1)$, $1\leqq A\leqq e$, in Corollary 2 is of the right order of n since the function $(1-z^2)^{-n}$ has for its derivative the function $2nz(1-z^2)^{-n-1}$. Indeed we have $2-2(n+1)^{-1}\leqq A\leqq e$.

In the particular case $n=1$, $C(1)=4$ and we have

$$|f'(z)|+|f(z)|^2\leqq K(1-|z|^2)^{-2},\qquad |z|<1, \tag{2.11}$$

with $K\leqq 4$. If $f(z)=(1-z^2)^{-1}$ and if $z=r$, $0<r<1$, we have

$$|f'(r)|+|f(r)|^2=(2r+1)(1-r^2)^{-2}\leqq 3(1-r^2)^{-2}$$

and we conclude that the smallest constant K in Corollary 3 is at least as large as 3. Thus $3\leqq K\leqq 4$.

As an application of Corollary 3 one can easily show that if $f(z)$ is analytic in E and satisfies the inequality

$$|f''(z)/f'(z)|\leqq \tfrac{1}{2}(1-|z|^2)^{-1},\qquad z\in E, \tag{2.12}$$

then the Schwarzian derivative $\omega(f,z)$ satisfies the inequality

$$|\omega(f,z)|=\left|\left(\frac{f''(z)}{f'(z)}\right)'-\frac{1}{2}\left(\frac{f''(z)}{f'(z)}\right)^2\right|\leqq 2(1-|z|^2)^{-2} \tag{2.13}$$

for $z\in E$, and consequently, by Nehari's Test [3], $f(z)$ is univalent for $|z|<1$. This is a slight improvement of a similar result given in [2] where the constant $2(\sqrt{5}-2)$ in (2.12), instead of the larger constant $1/2$, was found to be sufficient for univalency. The method used here is elementary and probably the factor $1/2$ in (2.12) can be replaced by a larger one and still the inequality (2.12) would force $f(z)$ to be univalent in E.

References

1. W. M. Causey, *The close-to-convexity and univalence of an integral*, Math. Z. **99** (1967), 207–212. MR **35** #6807.

2. P. L. Duren, H. S. Shapiro and A. L. Shields, *Singular measures and domains not of the Smirnov type*, Duke Math. J. **33** (1966), 247–254. MR **33** #7506.

3. Z. Nehari, *The Schwarzian derivative and schlicht functions*, Bull. Amer. Math. Soc. **55** (1949), 545–551. MR **10**, 696.

4. M. Nunokawa, *On the univalency and multivalency of certain analytic functions*, Math. Z. **104** (1968), 394–404. MR **37** #2971.

5. M. Zeigler, *A class of regular functions related to univalent functions*, Ph.D. dissertation, University of Delaware, Newark, Del., 1970.

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