PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 28, Number 2, May 1971

# OSCILLATION PROPERTIES OF THE 2-2 DISCONJUGATE FOURTH ORDER SELFADJOINT DIFFERENTIAL EQUATION

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ABSTRACT. This paper contains a proof that either all, or none, of the nontrivial solutions of the fourth order linear selfadjoint differential equation have an infinite number of zeros on a half line, provided that no nontrivial solution has more than one double zero on that half line.

Throughout this paper, let $Ly=(ry'')''-(qy')'+py$ where $r$, $q$, and $p$ are given real-valued functions, $a\in(-\infty,\ \infty)$ is given, $r''$, $q'$, $p\in C[a,\ \infty)$, and $r(t)>0$ for $t\geqq a$. A nontrivial solution to $Ly=0$ is said to oscillate if its zeros in $[a,\ \infty)$ are unbounded.

THEOREM 1. *If no nontrivial solution to $Ly=0$ has more than one double zero in $[a,\ \infty)$, then all the nontrivial solutions oscillate or none oscillate.*

W. Leighton and Z. Nehari [**1**, p. 367] obtain the same conclusion using the hypothesis that $q(t)\equiv 0$, $p(t)>0$ for $t\geqq a$. As they note, these assumptions imply that no nontrivial solution has more than one double zero. Some lemmas will be established before proving Theorem 1.

$Ly=0$ will be said to be 2-2 disconjugate if no nontrivial solution has more than one double zero in $[a,\ \infty)$. Of course, this is a special case of the concept known as $n$-$n$ disconjugacy. When $r$, $q$, and $p$ are all constants, it can easily be shown that $Ly=0$ is 2-2 disconjugate if and only if $rw^4+qw^2+p\geqq 0$ for all real numbers, $w$.

For $i=1, 2, 3$ and $a\leqq b<\infty$, let $y_{bi}$ designate the solution to

$$Ly=0,\quad y^{(j)}(b)=\delta_{ij},\qquad j=0,1,2,3,$$

where $\delta_{ij}$ is the Kronecker delta. Denote the zeros of $y_{b3}$ by

$$\cdots<\eta(b,-1)<b<\eta(b,1)<\eta(b,2)<\cdots$$

Received by the editors July 16, 1970.

*AMS* 1969 *subject classifications*. Primary 3420, 3442.

*Key words and phrases*. Conjugate point, disconjugacy, $n$-$n$ disconjugacy for $n=2$, Morse index, nonoscillatory solution, oscillatory solution, separation of zeros, selfadjoint differential equation of fourth order.

[1] The author is grateful to Professor A. C. Lazer for suggesting the problem of proving Theorem 1, for suggesting the short proof of Lemma 4, and for introducing the author to the conjugate point theory of M. Morse from which Lemma 8 is derived.

continuing in both directions until all the zeros of $y_{b3}$ in $[a, \infty)$ are named.

Let $W(u_1, \cdots, u_n) = \det(u_i^{(j-1)})$, $1 \leqq i, j \leqq n$, denote the Wronskian determinant of $u_1, \cdots, u_n$.

LEMMA 1. *If $Ly=0$ is 2-2 disconjugate, then for all $k$, $\eta(b, k)$ varies continuously with $b$.*

LEMMA 2. *If $Ly=0$ is 2-2 disconjugate and $u$ and $v$ are independent solutions with double zeros at $b \geqq a$, then $W(u, v)$ vanishes only at $b$.*

LEMMA 3. *If $W(u, v)$ never vanishes on an interval, then the zeros of $u$ and $v$ separate on that interval.*

The same proof given by Leighton and Nehari [1, p. 360] for the continuity of $\eta(b, k)$ proves Lemma 1. If, in Lemma 2, $W(u, v)(c) = 0$ for $c \neq b$, then some nontrivial linear combination of $u$ and $v$ has double zeros at both $b$ and $c$, contradicting 2-2 disconjugacy. Lemma 3 is stated by Leighton and Nehari [1, p. 327].

COROLLARY. *If $Ly=0$ is 2-2 disconjugate and $u$ and $v$ are independent solutions with double zeros at $b$, then the zeros of $u$ and $v$ separate on $(b, \infty)$, and on $[a, b)$ when $a<b$. Furthermore, if $v=y_{b3}$ and $\eta(b, 1)$ [resp. $\eta(b, -1)$] exists, then $u$ has a zero in $(b, \eta(b, 1))$ [resp. $(\eta(b, -1), b)$].*

PROOF. In light of Lemmas 2 and 3, only the last statement requires a proof. Let $c=\eta(b, 1)$. Assume, without loss in generality, that $u$ is positive in $(b, c]$. Then, for every constant $A>0$, there exists $\epsilon>0$ so $(Au-v)(t)>0$ for $b<t<b+\epsilon$. Since $v$ is uniformly bounded in $[b, c]$, there exists $A>0$ so $w=Au-v$ is nonnegative in $[b, c]$ and has a double zero in $(b, c)$. But $w$ is nontrivial and also has a double zero at $b$, contradicting 2-2 disconjugacy. The proof that $u$ has a zero in $(\eta(b, -1), b)$ is parallel.

If $Lu=0$, $u(b)=0\neq u'(b)$, then differentiation of

$$\begin{vmatrix} y_{b3} & y_{b2} & u \\ y_{b3}' & y_{b2}' & u' \\ ry_{b3}'' & ry_{b2}'' & ru'' \end{vmatrix}$$

four times shows it to be a nontrivial solution to $Ly=0$ with a triple zero at $b$, so $rW(y_{b3}, y_{b2}, u) = By_{b3}$ for some constant $B\neq 0$. This curious fact is an aid in the proof of Lemmas 4 and 5.

LEMMA 4. *If $b>a$ and $Ly=0$ is 2-2 disconjugate with solution $u$ in-*

*dependent of* $y_{b3}$ *such that* $u(b)=0$, *then* $W(y_{b3}, u)$ *has at most one zero in* $(b, \infty)$.

PROOF. Because of Lemma 2, suppose $u'(b)\neq 0$. By Lemma 2, $W(y_{b3}, y_{b2})(t)\neq 0$ for $t>b$. Let

$$f(t) = W(y_{b3}, u)(t)/W(y_{b3}, y_{b2})(t) \qquad \text{for } t > b.$$

Then

$$f' = \frac{y_{b3}W(y_{b3}, y_{b2}, u)}{(W(y_{b3}, y_{b2}))^2} = \frac{By_{b3}^2}{r(W(y_{b3}, y_{b2}))^2}\,.$$

The first equality is due to Pólya [**3**, p. 315], the second equality follows since $rW(y_{b3}, y_{b2}, u)=By_{b3}$ for some $B\neq 0$. This implies $f'$ cannot change sign in $(b, \infty)$ and is zero only on a discrete set of points. Thus $f$ has at most one zero in $(b, \infty)$. Consequently, $W(y_{b3}, u)$ has at most one zero there.

The following corollary is typical of the relationships which can be shown to hold between the zeros of nontrivial solutions to $Ly=0$ with simultaneous zeros. Extensions of these relations to pairs of arbitrary solutions follows from Lemma 7.

COROLLARY. *Let* $Ly=0$ *be* 2-2 *disconjugate with nontrivial solutions* $u$ *and* $v$ *such that* $u(b)=0=v(b)$ *for some* $b\geqq a$. *For* $a\leqq d<e<\infty$, *let* $N(u, d, e)$ *denote the number of zeros of* $u$ *in* $(d, e)$. *Then there exists* $c\geqq b$ *such that*

$$\left| N(u, d, e) - N(v, d, e)\right| \leqq 2 \qquad \textit{when } c \leqq d < e < \infty.$$

PROOF. By Lemma 4 there exists $c \geqq b$ so $W(y_{b3}, u)(t) \neq 0 \neq W(y_{b3}, v)(t)$ for $t>c$. By Lemma 3,

$$\begin{aligned}\left| N(u, d, e) - N(v, d, e)\right| &\leqq \left| N(y_{b3}, d, e) - N(u, d, e)\right| \\ &\quad + \left| N(y_{b3}, d, e) - N(v, d, e)\right| \\ &\leqq 1 + 1.\end{aligned}$$

LEMMA 5. *If* $Ly=0$ *is* 2-2 *disconjugate,* $b\neq c$, *and* $y_{c3}(b)=0$, *then* $c=\eta(b, k)$ *for some* $k$. *Furthermore, if* $b<c$, *then* $y_{b3}$ *and* $y_{c3}$ *both have* $k-1$ *zeros on* $(b, c)$. *When* $k>1$, *the zeros of* $y_{b3}$ *and* $y_{c3}$ *separate on* $(b, c)$ *with* $\eta(c, 1-k)<\eta(b, 1)$ *and* $\eta(c, -1)<\eta(b, k-1)$.

PROOF. Since $y_{c3}(b)=0=y_{c3}(c)=y'_{c3}(c)=y''_{c3}(c)$, $y_{c3}$ is a nontrivial linear combination of $y_{b3}$, $y_{b2}$, and $y_{b1}$, and $W(y_{b3}, y_{b2}, y_{b1})(c)=0$. Now $rW(y_{b3}, y_{b2}, y_{b1})=By_{b3}$ for $B\neq 0$, so $c=\eta(b, k)$ for some $k$. The zeros of $y_{b3}$ and $y_{c3}$ separate in $(b, c)$ if $b<c$, since $c$ is the only zero of

$W(y_{b3}, y_{c3})$ in $(b, \infty)$ by Lemma 4. Assume $k>1$ and $\eta(b, 1)<\eta(c, 1-k)$. Then for some constant $A$, $Ay_{c3}-y_{b3}$ has a double zero at some $d \in (b, \eta(b, 1))$, contradicting the nonvanishing property of $W(y_{b3}, y_{c3})$ on $(b, c)$. The proof that $\eta(c, -1)<\eta(b, k-1)$ is similar.

COROLLARY. *If $Ly=0$ is 2-2 disconjugate, then each $\eta(b, k)$ is a strictly increasing function of $b$.*

PROOF. By Lemma 1, if $\eta(b, k)$ were not strictly increasing, there would exist $c<d$ so $\eta(c, k)=\eta(d, k)=e$. Suppose $k>0$. By Lemma 5, $y_{e3}$ has $k-1$ zeros in both $(d, e)$ and $(c, e)$, contradicting the fact that $y_{e3}(d)=0$. If $k<0$ the proof is similar.

LEMMA 6. *If $Ly=0$ is 2-2 disconjugate, $b \geqq a$, and $\eta(b, 2)$ exists, then every solution has at least one zero in $(b, \eta(b, 2))$.*

PROOF. Let $c=\eta(b, 2)$. Let $u$ be a nontrivial solution to $Ly=0$. By Lemma 5 and the corollary to Lemma 3, $u$ has a zero in $[\eta(c, -1), c) \subset (b, c)$ if $u$ has a double zero at $c$. Assume $u$ is independent of $y_{b3}$ and, without loss in generality, is negative in $[\eta(b, 1), c)$. Two cases will be discussed separately.

*Case* 1. $u(c)=0<u'(c)$. Since $w(t)=u'(c)y_{b3}(t)-y_{b3}'(c)u(t)$ is a nontrivial solution and has a double zero at $c$, $w$ has exactly two simple zeros, $d$ and $e$, in $[b, c)$ with $b<d<\eta(c, -1)<e<c$ by the corollary to Lemma 3. If $e<\eta(b, 1)<c$, then $w$ is positive in $(e, c)$ since $w(\eta(b, 1)) = -y_{b3}'(c)u(\eta(b, 1))>0$. Hence

$$0 > w(\eta(c, -1)) = u'(c)y_{b3}(\eta(c, -1)) - y_{b3}'(c)u(\eta(c, -1)).$$

Now $u(\eta(c, -1))$ is positive because the other three terms are all positive, so $u$ has a zero in $(\eta(c, -1), \eta(b, 1)) \subset (b, c)$. If $\eta(c, -1) <\eta(b, 1)<e$, a similar proof can be given.

*Case* 2. $u(c)<0$. Since $u$ is negative on $[\eta(b, 1), c]$, for some constant $A>0$, $w=y_{b3}-Au$ has a double zero at some $d \in (\eta(b, 1), c)$ and is positive at all other points of this interval. By the corollary to Lemma 3,

$$0 > w(\eta(d, -1)) = y_{b3}(\eta(d, -1)) - Au(\eta(d, -1)).$$

Now $\eta(c, -1)<\eta(b, 1)$ by Lemma 5, and $\eta(b, 1)<d<c$, so $b<\eta(d, -1) <\eta(c, -1)<\eta(b, 1)$. Therefore $y_{b3}(\eta(d, -1))>0$, and thus $u(\eta(d, -1)) >0$. Hence $u$ has a zero in $(\eta(d, -1), \eta(b, 1)) \subset (b, c)$.

LEMMA 7. *If $Ly=0$ is 2-2 disconjugate with nontrivial solutions $u$ and $v$, and $u$ has five distinct zeros in $[b, c] \subset [a, \infty)$, then $v$ has at least one zero in $(b, c)$.*

PROOF. Suppose $b$ is the first of the five distinct zeros in $[b, c]$. By Lemma 4, $W(y_{b3}, u)$ has at most one zero in $(b, \eta(b, 2)]$. Therefore, by Lemma 3, $u$ has at most three distinct zeros in $(b, \eta(b, 2)]$, so $\eta(b, 2) < c$. By Lemma 6, $v$ has at least one zero in $(b, \eta(b, 2))$, so, a fortiori, $v$ has a zero in $(b, c)$.

PROOF OF THEOREM 1. If $y_{a3}$ oscillates and $Lu = 0$, then $u$ has a zero in $(\eta(a, 4j), \eta(a, 4j+4))$ for every positive integer $j$ by Lemma 7. If $y_{a3}$ has a largest zero $b$ and $u$ is a nontrivial solution to $Ly = 0$, then, by Lemma 7, $u$ can have no more than four zeros in $[b, \infty)$.

That the converse of Theorem 1 is not true follows from the fact that any nontrivial solution to

$$y^{(4)} + 5y'' + 4y = 0$$

oscillates, but $u(t) = \sin 2t - 2 \sin t$ is a solution with an infinite number of double zeros. Theorem 1 cannot be extended to $n$-$n$ disconjugate selfadjoint equations of order $2n$ for $n > 2$ since both $u(t) = t$ and $v(t) = \sin(t) \sinh(t)$ are solutions to $y^{(2n)} + 4y^{(2n-4)} = 0$ for $n > 2$. That Theorem 1 cannot be extended to include selfadjoint equations for which no solution can have more than two double zeros follows from $y^{(4)} - y = 0$. One possible extension of Theorem 1 is stated in Theorem 2.

Let $a_k$ for $k = 0, 1, \cdots, n$ be real-valued, sufficiently differentiable functions defined on $[a, \infty)$ with $a_n(t) > 0$ for $t \geqq a$. For $a \leqq b$, $c < \infty$, define $b$ to be $n$-$n$ conjugate to $c$ with multiplicity $m$ if

$$\sum_{k=0}^{n} (a_k y^{(k)})^{(k)} = 0, \qquad y^{(j)}(b) = 0 = y^{(j)}(c) \quad \text{for } j = 0, 1, \cdots, n-1$$

has $m$ independent solutions. A straightforward modification of the proofs of the Index and Separation Theorems of M. Morse [**2**, p. 7–18] yields the following lemma.

LEMMA 8. *For $b \geqq a$ and $a \leqq d < e < \infty$, let $m_n(b, (d, e))$ denote the sum of the multiplicities of the points $c \in (d, e)$ which are $n$-$n$ conjugate to $b$. Then*

$$\left| m_n(b, (d, e)) - m_n(b', (d, e)) \right| \leqq n$$

*when $a \leqq b, b', d < \infty; d < e < \infty$.*

THEOREM 2. *If $Ly = 0$ and $m_2(a, (a, e))$ is bounded for $e > a$, then either all, or none, of the nontrivial solutions oscillate.*

PROOF. Choose $b \geqq a$ so $m_2(a, (b, e)) = 0$ for all $e > b$, and choose $c > b$. By Lemma 8, $m_2(c, (b, e)) \leqq 2$ for all $e > b$. Since $c$ is 2-2 conjugate to itself with multiplicity 2, there exists no $c' > c$ such that $c'$ is 2-2 conjugate to $c$. Therefore $Ly = 0$ is 2-2 disconjugate on $[c, \infty)$, and Theorem 2 follows from Theorem 1.

ADDED IN PROOF. Lemma 5 and its corollary also appear in a recent paper by A. C. Peterson [**4**, pp. 505–506].

## REFERENCES

1. W. Leighton and Z. Nehari, *On the oscillation of solutions of self-adjoint linear differential equations of the fourth order*, Trans. Amer. Math. Soc. **89** (1958), 325–377. MR **21** #1429.

2. M. Morse, *Introduction to analysis in the large*, 2nd ed., Lectures, Institute for Advanced Study, Princeton, N.J., 1951; reprint, 1957. MR **16**, 837.

3. G. Pólya, *On the mean-value theorem corresponding to a given linear homogeneous differential equation*, Trans. Amer. Math. Soc. **24** (1922), 312–324.

4. A. Peterson, *The distribution of zeros of extremal solutions of a fourth order differential equation for the n-th conjugate point*, J. Differential Equations **8** (1970), 502–511.

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