PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 28, Number 2, May 1971

IDEMPOTENTS IN GROUP RINGS

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ABSTRACT. In this note we offer an elementary entirely self-contained proof of a theorem of Kaplansky on idempotents in complex group rings.

Let $C[G]$ denote the (discrete) group algebra of a not necessarily finite group G over the complex numbers. If

$$\alpha = \sum_{x \in G} a_x \cdot x \in C[G]$$

then we define the trace of α to be $\operatorname{tr} \alpha = a_1$, the coefficient of $1 \in G$.

THEOREM (KAPLANSKY). *Let $e \neq 0$, 1 be an idempotent in $C[G]$. Then* tr *e is a totally real algebraic number with the property that it and all its conjugates lie strictly between* 0 *and* 1.

In the original proof of this result (see [1, pp. 122–123]), $C[G]$ was embedded into $W(G)$, the weak closure of its action on the Hilbert space $L_2(G)$. In a later proof [2] due to Susan Montgomery, $C[G]$ was embedded into the uniform closure of its action on $L_2(G)$. In this note we offer an elementary, completely self-contained proof of this result and we work entirely within $C[G]$.

Let $\alpha = \sum a_x \cdot x$, $\beta = \sum b_x \cdot x$, $\gamma = \sum c_x \cdot x$ be elements of $C[G]$. We define an inner product and appropriate norms on $C[G]$ by

$$(\alpha, \beta) = \sum_x a_x \bar{b}_x,$$

$$\|\alpha\| = (\alpha, \alpha)^{1/2} = \left(\sum_x |a_x|^2\right)^{1/2},$$

$$|\alpha| = \sum_x |a_x|,$$

where $^-$ denotes complex conjugation.

LEMMA 1. *With the above notation we have*
(i) $\operatorname{tr} \alpha\beta = \operatorname{tr} \beta\alpha$;
(ii) $\|\alpha + \beta\| \leq \|\alpha\| + \|\beta\|$;
(iii) $|\operatorname{tr} \alpha| \leq \|\alpha\|$, $(\alpha, 1) = \operatorname{tr} \alpha$;

Received by the editors June 29, 1970.
AMS 1969 *subject classifications*. Primary 2080.
Key words and phrases. Complex group rings, idempotents, traces.

(iv) $(\alpha, \beta\gamma) = (\alpha\gamma^*, \beta)$ *where* $\gamma^* = \sum \bar{c}_x \cdot x^{-1}$;
(v) $\|\alpha\beta\| \leqq \|\alpha\| \cdot |\beta|$.

PROOF. Clearly $\operatorname{tr} \alpha\beta = \sum_x a_x b_{x^{-1}}$ and this is symmetric in the a's and b's so (i) follows. Part (ii) is of course just the triangle inequality and (iii) is obvious.

Now it is easy to see that the map $*:\gamma \to \gamma^*$ is in fact an anti-automorphism of $C[G]$. Moreover

$$(\alpha, \beta) = \sum_x a_x \bar{b}_x = \operatorname{tr} \alpha\beta^*.$$

Thus

$$(\alpha, \beta\gamma) = \operatorname{tr} \alpha(\beta\gamma)^* = \operatorname{tr} \alpha\gamma^*\beta^* = (\alpha\gamma^*, \beta)$$

and (iv) is proved. As a special case of this, we observe that for $x \in G$

$$\|\alpha x\| = (\alpha x, \alpha x)^{1/2} = (\alpha x x^*, \alpha)^{1/2} = (\alpha, \alpha)^{1/2} = \|\alpha\|$$

since $x^* = x^{-1}$.

Finally by (ii) and the above

$$\|\alpha\beta\| = \left\|\sum_x \alpha b_x \cdot x\right\| \leqq \sum_x \|\alpha b_x \cdot x\| = \sum_x \|\alpha\| \, |b_x| = \|\alpha\| \cdot |\beta|$$

and the result follows.

Now let $e \neq 0$ be an idempotent in $C[G]$ and set $M = eC[G]$ and $d = \inf_{\alpha \in M} \|1 - \alpha\|$. For each integer n choose $\alpha_n \in M$ with $\|1 - \alpha_n\|^2 < d^2 + 1/n^4$.

The following lemma is the key to our proof of the theorem.

LEMMA 2. *Let* $\beta \in M$. *Then*

$$|(\beta, 1 - \alpha_n)| \leqq \|\beta\|/n^2.$$

PROOF. This is trivial for $\beta = 0$ so assume $\beta \neq 0$ and set $k = (1 - \alpha_n, \beta)/\|\beta\|^2$. Then $\alpha_n + k\beta \in M$ so

$$\|1 - \alpha_n - k\beta\|^2 \geqq d^2 > \|1 - \alpha_n\|^2 - 1/n^4.$$

Thus

$$\begin{aligned} 1/n^4 &> \|1 - \alpha_n\|^2 - \|1 - \alpha_n - k\beta\|^2 \\ &= (1 - \alpha_n, 1 - \alpha_n) - (1 - \alpha_n - k\beta, 1 - \alpha_n - k\beta) \\ &= k(\beta, 1 - \alpha_n) + \bar{k}(1 - \alpha_n, \beta) - k\bar{k}(\beta, \beta) \\ &= |(\beta, 1 - \alpha_n)|^2/\|\beta\|^2 \end{aligned}$$

and the result follows.

LEMMA 3. *There exists nonnegative real constants r' and r'' with*
(i) $|\operatorname{tr}\alpha_n - \|\alpha_n\|^2| \leqq r'/n$,
(ii) $\|e-\alpha_n e\| \leqq r''/n$.

PROOF. We first observe that

$$\|\alpha_n - 1\| \leqq (d^2 + 1/n^4)^{1/2} \leqq d + 1$$

and, by Lemma 1(ii),

$$\|\alpha_n\| \leqq \|1\| + \|\alpha_n - 1\| \leqq d + 2.$$

(i). Since $\alpha_n \in M$, Lemma 2 yields

$$|(\alpha_n, 1 - \alpha_n)| \leqq \|\alpha_n\|/n^2 \leqq \|\alpha_n\|/n \leqq (d + 2)/n.$$

Moreover, by Lemma 1(iii), we have

$$(\alpha_n, 1 - \alpha_n) = (\alpha_n, 1) - (\alpha_n, \alpha_n) = \operatorname{tr}\alpha_n - \|\alpha_n\|^2$$

so (i) follows with $r' = d+2$.
(ii) By Lemma 1(iv)

$$\begin{aligned}\|e - \alpha_n e\|^2 &= ((1 - \alpha_n)e, (1 - \alpha_n)e) \\ &= ((1 - \alpha_n)ee^*, 1 - \alpha_n).\end{aligned}$$

Now $(1-\alpha_n)ee^* = ee^* - \alpha_n ee^* \in M$ so Lemma 1(v) and Lemma 2 yield

$$\begin{aligned}\|e - \alpha_n e\|^2 &\leqq \|(1 - \alpha_n)ee^*\|/n^2 \\ &\leqq \|1 - \alpha_n\| \cdot |ee^*|/n^2 \\ &\leqq (d + 1) \cdot |ee^*|/n^2.\end{aligned}$$

Thus the result follows with $(r'')^2 = (d+1) \cdot |ee^*|$.

LEMMA 4. $\operatorname{tr} e$ *is real and* $\operatorname{tr} e \geqq \|e\|^2/|e|^2 > 0$.

PROOF. By Lemma 1(iii) and Lemma 3 we have

$$\begin{aligned}|\operatorname{tr}\alpha_n - \|\alpha_n\|^2| &\leqq r'/n, \\ |\operatorname{tr} e - \operatorname{tr}\alpha_n e| &\leqq \|e - \alpha_n e\| \leqq r''/n.\end{aligned}$$

Moreover, by Lemma 1(i), $\operatorname{tr}\alpha_n e = \operatorname{tr} e\alpha_n = \operatorname{tr}\alpha_n$ since $\alpha_n \in M$ implies that $e\alpha_n = \alpha_n$. Thus we have from the above

$$|\operatorname{tr} e - \|\alpha_n\|^2| \leqq (r' + r'')/n$$

and we conclude that

$$\operatorname{tr} e = \lim_{n\to\infty} \|\alpha_n\|^2.$$

Therefore $\operatorname{tr} e$ is real and nonnegative.

Now Lemma 1(ii), (v) and Lemma 3(ii) yield

$$\|e\| \leqq \|e - \alpha_n e\| + \|\alpha_n e\| \leqq r''/n + \|\alpha_n\| \cdot |e|.$$

Thus taking limits as $n \to \infty$ we obtain

$$\|e\| \leqq (\mathrm{tr}\, e)^{1/2} \cdot |e| \quad \text{and} \quad \mathrm{tr}\, e \geqq \|e\|^2 / |e|^2 > 0.$$

We now proceed to prove the Theorem. Let $e \neq 0$, 1 be an idempotent in $C[G]$. Then by Lemma 4, tr $e > 0$. Since $1-e$ is also a nonzero idempotent, Lemma 4 yields $1 - \mathrm{tr}\, e = \mathrm{tr}(1-e) > 0$ so tr $e < 1$. Let σ be a field automorphism of the complex numbers. Then σ clearly induces a ring automorphism of $C[G]$ by

$$\alpha^{\sigma} = \sum_x a_x^{\sigma} \cdot x.$$

Since e^{σ} is again an idempotent in $C[G]$ and tr $e^{\sigma} = (\mathrm{tr}\, e)^{\sigma}$ the above yields $1 > (\mathrm{tr}\, e)^{\sigma} > 0$ for all σ. Now if tr e is transcendental over the rationals then there certainly exists a field automorphism σ such that $(\mathrm{tr}\, e)^{\sigma}$ is not real, a contradiction. Thus tr e is algebraic and the Theorem is proved.

There is an interesting consequence of this result, again due to Kaplansky, which we include for the sake of completeness.

COROLLARY. *Let α, $\beta \in C[G]$ and suppose that $\alpha\beta = 1$. Then $\beta\alpha = 1$.*

PROOF. Set $e = \beta\alpha$. Then

$$e^2 = \beta(\alpha\beta)\alpha = \beta\alpha = e$$

so e is an idempotent in $C[G]$. Moreover, by Lemma 1(i),

$$\mathrm{tr}\, e = \mathrm{tr}\, \beta\alpha = \mathrm{tr}\, \alpha\beta = 1.$$

Thus by the Theorem we must have $e = 1$.

REFERENCES

1. I. Kaplansky, *Fields and rings*, Chicago Lectures in Math., Univ. of Chicago Press, Chicago, Ill., 1969.

2. M. S. Montgomery, *Left and right inverses in group algebras*, Bull. Amer. Math. Soc. **75** (1969), 539–540. MR **39** #327.

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