PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 28, Number 2, May 1971

STABLE HOMEOMORPHISMS ON INFINITE-DIMENSIONAL NORMED LINEAR SPACES

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ABSTRACT. R. Y. T. Wong has recently shown that all homeomorphisms on a connected manifold modeled on infinite-dimensional separable Hilbert space are stable. In this paper we establish the stability of all homeomorphisms on a normed linear space E such that E is homeomorphic to the countable infinite product of copies of itself. The relationship between stability of homeomorphisms and a strong annulus conjecture is demonstrated and used to show that stability of all homeomorphisms on a normed linear space E implies stability of all homeomorphisms on a connected manifold modeled on E, and that in such a manifold collared E-cells are tame.

1. **Introduction.** R. Y. T. Wong showed in [12] that all homeomorphisms on s, the countable infinite product of lines, are stable. He later showed [13] that all homeomorphisms on a connected manifold modeled on s are stable. (It is known that s is homeomorphic to every infinite-dimensional separable Fréchet space.) In this paper we extend the first result to normed linear spaces E such that E is homeomorphic to E^ω, the countable infinite product of copies of E. Such spaces include all infinite-dimensional Hilbert and reflexive Banach spaces [2]. Presently, there are no known examples of infinite-dimensional Banach spaces which do not satisfy this condition. We also establish, by an argument more general than that in [13], the stability of all homeomorphisms on a connected manifold modeled on a normed linear space E such that all homeomorphisms on E are stable.

2. **Stability of homeomorphisms.** For a topological space X, $H(X)$ will denote the group of homeomorphisms, and $SH(X)$ the subgroup of stable homeomorphisms.

2.1. LEMMA (KLEE [8]). *For E an infinite-dimensional normed linear space, there exists a homeomorphism of E onto $E\setminus\{0\}$, supported on the unit ball.*

Presented to the Society, February 6, 1970; received by the editors January 31, 1970 and, in revised form, June 23, 1970.

AMS 1969 *subject classifications*. Primary 4610, 5755.

Key words and phrases. Infinite-dimensional normed linear spaces, connected infinite-dimensional manifolds, stable homeomorphisms, collared cells, annulus conjecture.

2.2. THEOREM. *If E is a normed linear space such that $E \simeq E^\omega$, then $SH(E) = H(E)$.*

PROOF. By (2.1) we have $E \simeq S_1 \times (-1, 1)$, where S_1 is the unit sphere. Thus $E \simeq E^\omega \simeq S_1^\omega \times s \simeq S_1^\omega \times s \times [0, 1) \simeq E \times [0, 1)$. Let $h \in H(E \times [0, 1))$ and $K = E \times \{0\}$. With the above hypotheses on E, Cutler [5] has shown that K is E-deficient (i.e., there exists a homeomorphism of $E \times [0, 1)$ onto $E \times E$ taking K into $E \times \{0\}$), and that the finite union of E-deficient sets is E-deficient. Thus $K \cup h(K) \subset E \times [0, 1)$ is E-deficient, and we may consider $E \times [0, 1)$ as $E \times E$, with $K \cup h(K) \subset E \times \{0\}$. Rather standard techniques yield a stable homeomorphism $g \in SH(E \times E)$ such that $g/K = h/K$. (Let $j: K \to K'$ be a homeomorphism onto a closed copy of K properly contained in $\{0\} \times E$. By use of Dugundji's extension theorem [6], the projections of the graphs of j and $jh^{-1}/h(K)$ into $E \times \{0\}$ and $\{0\} \times E$ can be extended by stable homeomorphisms of $E \times E$. See Klee [7], Anderson [1].) Then $g^{-1}h/K = \mathrm{id}$, and an easy argument due to Wong [12] shows that $g^{-1}h$ is stable. Thus $h = g(g^{-1}h)$ is stable.

3. Stable homeomorphisms and the strong annulus conjecture.

For E a normed linear space, B_r will denote the closed ball with radius r centered at the origin, and S_r the boundary sphere. A subset C of a space X is an *E-cell* if there exists a homeomorphism from (B_1, S_1) onto $(C, \mathrm{Bd}\, C)$, where $\mathrm{Bd}\, C$ is the topological boundary of C in X. C is a *collared E-cell* if, for some subset C' of X, there exists a homeomorphism from $(B_2; B_1, S_2)$ onto $(C'; C, \mathrm{Bd}\, C')$. A subset A of X is an *E-annulus* if there exists a homeomorphism from $(B_2 \backslash \mathrm{Int}\, B_1, S_1 \cup S_2)$ onto $(A, \mathrm{Bd}\, A)$. The *annulus conjecture* for E (which we denote by $\mathrm{Ann}(E)$) says that for a collared cell $C \subset \mathrm{Int}\, B_2$, there exists a homeomorphism h from $(B_2 \backslash \mathrm{Int}\, B_1; S_1, S_2)$ onto $(B_2 \backslash \mathrm{Int}\, C; \mathrm{Bd}\, C, S_2)$. Clearly, we may specify that h/S_2 be the identity. Moreover, if we can simultaneously specify that $h/S_1 = f$, where f is any given homeomorphism from S_1 onto $\mathrm{Bd}\, C$, then we shall say that E satisfies the *strong annulus conjecture* ($\mathrm{S\,Ann}(E)$).

Two homeomorphisms $f, g \in H(X)$ are *weakly isotopic* if there exists a homeomorphism $H: X \times I \to X \times I$ such that $H(x, 0) = (f(x), 0)$ and $H(x, 1) = (g(x), 1)$ for every $x \in X$.

3.1. LEMMA. $\mathrm{S\,Ann}(E)$ *is equivalent to* $\mathrm{Ann}(E)$ *together with weak isotopy to the identity of homeomorphisms on the unit sphere* S_1.

The following Schoenflies theorem was proved by Brown [3] in the finite-dimensional case, and extended by Sanderson [10] to the infinite-dimensional case.

3.2. LEMMA. *Every collared cell in E is tame (i.e., there exists $h \in H(E)$ with $h(B_1) = C$).*

We now show that the stability of all homeomorphisms on an infinite-dimensional normed linear space E is equivalent to S Ann(E). This is the exact analogue of the results for Euclidean spaces (where only orientation-preserving homeomorphisms are considered), developed by Brown-Gluck [**4**]. A similar, but slightly different, infinite-dimensional result can be found in [**9**].

Let $BH(X)$ denote the subset of homeomorphisms of X which can be bridged to the identity; i.e., $f \in BH(X)$ if for every pair $\{x, y\} \subset X$ such that $x \neq y \neq f(x)$, there exist neighborhoods U of x and V of y, and $h \in H(X)$, such that $h/U = f/U$ and $h/V = \mathrm{id}$. This condition is a sharpened form of one employed by Whittaker [**11**], and the following lemma (which applies also to connected manifolds modeled on normed linear spaces) can be proved by using essentially his techniques.

3.3. LEMMA. *If E is a locally convex linear Hausdorff space with* $\dim E > 1$, *then $SH(E) = BH(E)$.*

3.4. LEMMA. *Let E be a normed linear space, and let C_1, C_2, and C_3 be E-cells such that $C_1 \subset \mathrm{Int}\ C_2 \subset C_2 \subset \mathrm{Int}\ C_3$, C_2 is collared, and $C_3 \backslash \mathrm{Int}\ C_1$ is an E-annulus. Then $C_2 \backslash \mathrm{Int}\ C_1$ is an E-annulus if and only if $C_3 \backslash \mathrm{Int}\ C_2$ is an E-annulus.*

PROOF. Standard techniques of Brown-Gluck for Euclidean spaces are applicable in the general case by virtue of the fact that every neighborhood of a collared cell contains a collar [**10**].

3.5. LEMMA. *Let E be a normed linear space, and let $h \in SH(E)$ with $h(B_1) \subset \mathrm{Int}\ B_2$. Then $B_2 \backslash \mathrm{Int}\ h(B_1)$ is an E-annulus.*

3.6. LEMMA. *Let E be a normed linear space, let $h \in H(S_1)$, and let $h^* \in H(E)$ be the radial extension of h. Then if $h^* \in SH(E)$, h is weakly isotopic to the identity.*

PROOFS. Again, these are generalizations of results of Brown-Gluck. We use the one-point cobounded extension $\tilde{E}$ of E, obtained by adjoining a point ω to E with neighborhoods of the form $\omega \cup U$, where $E \backslash U$ is closed and bounded. In the finite-dimensional case, E is the one-point compactification used by Brown-Gluck, and every homeomorphism of E can be trivially extended to a homeomorphism of $\tilde{E}$. In the infinite-dimensional case there exists, by Klee's result (2.1), a homeomorphism of E onto $\tilde{E}$ which is the identity on B_1. In (3.5) we may assume without loss of generality that $h(0)=0$, and then apply (3.3) and the above remark to obtain a homeomorphism $\tilde{h}$ of $\tilde{E}$ which agrees with h on a neighborhood of the origin and is the identity on a neighborhood of a point outside B_1. Clearly, we may assume that $\tilde{h}$ is the identity on a neighborhood of ω, and an application of (3.4) completes the proof. The proof of (3.6) is identical, with the last step unnecessary.

3.7. THEOREM. *Let E be an infinite-dimensional normed linear space. Then $SH(E)=H(E)$ if and only if* S Ann(E) *is true.*

PROOF. That $SH(E)=H(E)$ implies S Ann (E) follows from (3.1), (3.2), (3.5), and (3.6). The proof of the converse is contained in the proof of the following corollary.

3.8. COROLLARY. *If M is a connected manifold modeled on a normed linear space E, and $SH(E)=H(E)$, then $SH(M)=H(M)$.*

PROOF. Let $h\in H(M)$ and $p\in M$. Since there exists $g\in SH(M)$ with $gh(p)=p$, we may assume that $h(p)=p$. Let U be a neighborhood of p for which there exists a homeomorphism k from E into U such that $k(B_1)$ is closed in M and $k(0)=p$. Then $hk(B_\delta)\subset\mathrm{Int}\ k(B_1)$ for some δ, $0<\delta<1$. Let $C_1=k(B_\delta)$ and $C_2=k(B_1)$. By (3.7) there exists a homeomorphism f from $(C_2\backslash\mathrm{Int}\ C_1;\ \mathrm{Bd}\ C_1, \mathrm{Bd}\ C_2)$ onto $(C_2\backslash\mathrm{Int}\ h(C_1);\ \mathrm{Bd}\ h(C_1), \mathrm{Bd}\ C_2)$ such that $f/\mathrm{Bd}\ C_1=h/\mathrm{Bd}\ C_1$ and $f/\mathrm{Bd}\ C_2=\mathrm{id}$. Extending f by the identity outside C_2 and by h inside C_1, we have $h=f(f^{-1}h)\in SH(M)$.

3.9. COROLLARY. *If M is as above, and C_1 and C_2 are collared E-cells in M, there exists $h\in H(M)$ with $h(C_1)=C_2$.*

PROOF. Since M is connected, we may assume that $C_1\subset\mathrm{Int}\ C_2$. Let A be a collar of C_2. Then $C_2\cup A\backslash\mathrm{Int}\ C_1$ and $C_2\cup A\backslash\mathrm{Int}\ C_2$ are E-annuli, and there exists $h\in H(M)$ which is supported on $C_2\cup A$ and extends any given homeomorphism of $(C_1, \mathrm{Bd}\ C_1)$ onto $(C_2, \mathrm{Bd}\ C_2)$.

The *connected sum* $M_1 \# M_2$ of two connected manifolds modeled on a normed linear space E is obtained by deleting the interiors of collared E-cells in M_1 and M_2 and sewing together along the boundaries. By (3.9), if $SH(E)=H(E)$, then $M_1 \# M_2$ is a well-defined manifold modeled on E.

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