

ON THE SOLUTIONS OF A SEQUENCE OF LAMÉ DIFFERENTIAL EQUATIONS

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ABSTRACT. In a paper by H. Triebel, the solutions of some Lamé differential equations are interpreted in terms of conformal mappings. In the present paper, they are interpreted and constructed as covering projections from the unit disk onto Riemann surfaces with signature. Furthermore, the continuous dependence of solutions on the coefficient is established.

I. Introduction. We shall consider the following Lamé differential equations:

$$(1) \quad \eta''(z) + \left[\frac{1}{4}(1 - 1/n^2)\wp(z; 1, \tau) + C\right]\eta(z) = 0,$$

$$(2) \quad \eta''(z) + \left[\frac{1}{4}\wp(z; 1, \tau) + K\right]\eta(z) = 0,$$

where $\wp(z; 1, \tau)$ is the Weierstrass \wp -function with periods 1, τ , ($\tau = i|\tau|$, $|\tau| > 0$), C, K are real parameters, $n \geq 2$ is a positive integer. Triebel [4, §5] shows that for given n and τ as above, there exists a unique real number C (depending on η, τ), such that (1) possesses two linearly independent solutions. In fact, they can be constructed explicitly in terms of abelian integrals on a certain compact Riemann surface. But this construction is not valid for (2) because we no longer have a compact Riemann surface in this case.

In this paper, we shall consider the equivalent Schwarzian differential equations to (1) and (2), construct their solutions and interpret them as covering projections from the unit disk onto Riemann surfaces with signature. By a continuity theorem for Fuchsian groups (Wong [5]), we then conclude that for a fixed τ , the solutions of the Schwarzian differential equations corresponding to (1) tend to those corresponding to (2) as n tends to ∞ . The same result holds for (1) and (2). The main theorems are given in §IV.

II. Preliminaries. The following well-known result shows the equivalence of a Lamé differential equation to a Schwarzian differential equation.

THEOREM 1. *If $w(z)$ satisfies the Schwarzian differential equation:*

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$$(3) \quad \{w, z\} = \frac{1}{2}(1 - 1/n^2)\wp(z; 1, \tau) + 2C,$$

where τ , C , n are as above, then $1/(w'(z))^{1/2}$, $w(z)/(w'(z))^{1/2}$ are two linearly independent solutions of (1). Conversely, if $\eta(z)$ is a solution of (1), and $\eta(z) \neq 0$, then $w(z) = \int d\eta/\eta^2$ satisfies (3).

The general solution of (3) is given by $(aw+b)/(cw+d)$, a, b, c, d complex numbers, and $ad-bc \neq 0$.

The same result holds for (2) and the Schwarzian differential equation:

$$(4) \quad \{w, z\} = \frac{1}{2}\wp(z; 1, \tau) + 2K.$$

Here,

$$\{w, z\} = \frac{w'''}{w'} - \frac{3}{2} \left(\frac{w''}{w'} \right)^2 = \frac{d^2}{dz^2} \log w'(z) - \frac{1}{2} \left(\frac{d}{dz} \log w'(z) \right)^2$$

is the Schwarzian derivative.

Next, we shall state some definitions and facts about Fuchsian groups and Riemann surfaces with signature, which will be used in the coming sections. The general linear transformation $w = T(z) = (az+b)/(cz+d)$, $a, b, c, d \in \mathbb{C}$, and $ad-bc \neq 0$, is a conformal self-mapping of the Riemann sphere $\mathbb{C} \cup \{\infty\}$. Here \mathbb{C} denotes the complex plane. We shall normalize the transformation by requiring that $ad-bc=1$, and represent it by a matrix $T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Let $\mathfrak{X} = a+d$ be the trace of T . Then T is called elliptic, hyperbolic, or parabolic if \mathfrak{X} is real and $|\mathfrak{X}| < 2$, $|\mathfrak{X}| > 2$, or $|\mathfrak{X}| = 2$, respectively. T is called loxodromic if \mathfrak{X} is nonreal. If $T^m = \text{id}$ for some integer $m > 1$, then T is elliptic and if $n > 1$ is the smallest such integer, then T is said to be of order n . In this case $\mathfrak{X} = \pm 2 \cos(\pi/n)$, the converse is also true.

We shall consider only those linear transformations which leave the unit circle fixed and map the unit disk Δ onto itself. Then an elliptic transformation will have two fixed points: one inside Δ , the other being the inverse image of it with respect to the unit circle. A hyperbolic transformation will have two fixed points, both on the unit circle. A parabolic transformation will have one fixed point, which is on the unit circle.

Let G be a group of linear transformations from Δ onto itself. G is called properly discontinuous, if every point z_0 of Δ is contained in a neighborhood which contains only finitely many points of the orbit $\{Az_0 | A \in G\}$. In this case, G is also called a Fuchsian group.

Let S be a Riemann surface and $\{P_k\}$, $k=1, 2, 3, \dots$, be a discrete (finite or infinite) sequence of points on S . Let there be an "in-

teger" $\nu_k \geq 2$ associated with each point P_k (here ν_k may be an actual integer or ∞). This sequence of points and "integers" is called a "signature" on the Riemann surface and the triple $(S, \{P_k\}, \{\nu_k\})$ is called a Riemann surface with signature. Except for a few cases, a Riemann surface can always be represented by a Fuchsian group. More precisely, there exists a Fuchsian group G such that $S - \bigcup_{\nu_k=\infty} \{P_k\}$ is conformally equivalent to Δ/G . $S - \bigcup_{\nu_k \geq 2} \{P_k\}$ is conformally equivalent to Δ_G/G , where $\Delta_G = \Delta - \{\text{all elliptic fixed points of } G\}$, and S is conformally equivalent to $\hat{\Delta}_G/G$, where $\hat{\Delta}_G = \Delta \cup \{\text{all parabolic fixed points of } G\}$. The natural projection $\Delta \rightarrow \Delta/G$ followed by the conformal mapping $\Delta/G \rightarrow S - E_{\nu_k=\infty} \{P_k\}$ is locally 1 to 1 at each point of Δ_G , and is locally ν_k to 1 at the pre-images of P_k with $\nu_k < \infty$. G is determined uniquely up to conjugation by a linear transformation from Δ onto itself. This is the celebrated Koebe's theorem. Its statement can be found in Koebe [3], and a detailed proof is given in Wong [5].

If keeping the points P_k fixed, we vary the numbers ν_k in such a manner that the signature tends to a limit signature, and if we normalize the representing Fuchsian groups in question, then the corresponding (normalized) representing Fuchsian groups converge to the (normalized) representing Fuchsian group which corresponds to the limit signature. For a precise statement and proof, we refer to Wong [5]. This is the continuity theorem we shall use in the proof of part (ii) of Theorem 3, §IV. The meaning of the theorem will become clear there.

III. Construction of a family of Fuchsian groups. In this section, we shall construct a family of Fuchsian groups, from which we shall obtain a family of Riemann surfaces with signature. The corresponding covering projections will be solutions to (3) and (4). Consider the following linear transformations from Δ onto itself:

$$(5) \quad A = \frac{1}{(1 - \alpha^2)^{1/2}} \begin{pmatrix} 1 & \alpha \\ \alpha & 1 \end{pmatrix}, \quad 0 < \alpha < 1,$$

$$(6) \quad B = \frac{1}{(1 - \beta^2)^{1/2}} \begin{pmatrix} 1 & i\beta \\ -i\beta & 1 \end{pmatrix}, \quad 0 < \beta < 1.$$

They are both hyperbolic transformations. A has fixed points $-1, 1$ and $A(0) = \alpha$. B has fixed points $-i, i$ and $B(0) = i\beta$. $D = BAB^{-1}A^{-1}$ is a linear transformation from Δ onto itself with trace $D = 2 - 4\alpha^2\beta^2/(1 - \alpha^2)(1 - \beta^2)$. Therefore, if

$$(*) \quad \alpha^2\beta^2 = (1 - \alpha^2)(1 - \beta^2),$$

D is a parabolic transformation with fixed point

$$\delta^{(\infty)} = (\alpha + i\beta) / |\alpha + i\beta|.$$

On the other hand, if

$$(**) \quad 2\alpha^2\beta^2 = (1 - \alpha^2)(1 - \beta^2)(1 + \cos(\pi/n)), \quad n \geq 2 \text{ an integer,}$$

D is an elliptic transformation of order n . The fixed point of D inside Δ is given by

$$\delta^{(n)} = \frac{1}{\alpha - i\beta} \left[1 - \left(\alpha\beta \sin \frac{\pi}{n} \right) / \left(1 + \cos \frac{\pi}{n} \right) \right].$$

We shall denote the group generated by A, B with α, β satisfying (*) by $G_\alpha^{(\infty)}$. The group generated by A, B with α, β satisfying (**) by $G_\alpha^{(n)}$. Let

$$\begin{aligned} \delta_1^{(\infty)} &= -\overline{\delta^{(\infty)}}, & \delta_2^{(\infty)} &= -\delta^{(\infty)}, & \delta_3^{(\infty)} &= \overline{\delta^{(\infty)}}; \\ \delta_1^{(n)} &= -\overline{\delta^{(n)}}, & \delta_2^{(n)} &= -\delta^{(n)}, & \delta_3^{(n)} &= \overline{\delta^{(n)}}. \end{aligned}$$

Join $\delta^{(\infty)}, \delta_1^{(\infty)}, \delta_2^{(\infty)}, \delta_3^{(\infty)}$ with circular arcs perpendicular to the unit circle. Call the region bounded by them $R_\alpha^{(\infty)}$. Do the same for $\delta^{(n)}, \delta_1^{(n)}, \delta_2^{(n)}, \delta_3^{(n)}$ and get the region $R_\alpha^{(n)}$. We conclude easily that $A(\delta_1^{(\infty)}) = \delta^{(\infty)}, A(\delta_2^{(\infty)}) = \delta_3^{(\infty)}$ and A maps the arc $\delta_1^{(\infty)}\delta_2^{(\infty)}$ onto $\delta^{(\infty)}\delta_3^{(\infty)}$. Similarly, $B(\delta_2^{(\infty)}) = \delta_1^{(\infty)}, B(\delta_3^{(\infty)}) = \delta^{(\infty)}$ and B maps $\delta_2^{(\infty)}\delta_3^{(\infty)}$ onto $\delta_1^{(\infty)}\delta^{(\infty)}$. Furthermore, $R_\alpha^{(\infty)}$ is a fundamental region for the group $G_\alpha^{(\infty)}$. And the vertices $\delta^{(\infty)}, \delta_1^{(\infty)}, \delta_2^{(\infty)}, \delta_3^{(\infty)}$ form a parabolic cycle. Similarly, $R_\alpha^{(n)}$ is a fundamental region for the group $G_\alpha^{(n)}$. The vertices $\delta^{(n)}, \delta_1^{(n)}, \delta_2^{(n)}, \delta_3^{(n)}$ form an elliptic cycle with sum of angles of the vertices equal to $2\pi/n$. Therefore, each angle is exactly $\pi/2n$. From the existence of their fundamental regions, it follows that $G_\alpha^{(\infty)}, G_\alpha^{(n)}$ are properly discontinuous, hence Fuchsian groups. All these facts can be found in Ford [2] or any text on automorphic functions.

IV. Main theorems. For simplicity, if $\tau = i|\tau|$, $|\tau| > 0$, then let I_τ be the rectangle with vertices $0, 1, 1+\tau, \tau$. Let μ denote an integer ≥ 2 or ∞ .

THEOREM 2. (i) *For a fixed μ , given any $0 < \alpha < 1$, there exists a unique τ ($\tau = i|\tau|$, $|\tau| > 0$), and a unique conformal mapping $\phi_\alpha^{(\mu)}$ from $R_\alpha^{(\mu)}$ onto I_τ , such that $\phi_\alpha^{(\mu)}(\delta^{(\mu)}) = 1+\tau$, $\phi_\alpha^{(\mu)}(\delta_1^{(\mu)}) = \tau$, $\phi_\alpha^{(\mu)}(\delta_2^{(\mu)}) = 0$, $\phi_\alpha^{(\mu)}(\delta_3^{(\mu)}) = 1$. Write $\tau = \Phi^{(\mu)}(\alpha)$.*

(ii) *$\Phi^{(\mu)}$ is an 1-1, continuous, monotonic-decreasing function onto the positive imaginary axis, such that $|\tau| \rightarrow \infty$ as $\alpha \rightarrow 0$ and $|\tau| \rightarrow 0$ as $\alpha \rightarrow 1$.*

PROOF. (a) Let O be the center of Δ , P the intersection of $\delta^{(\mu)}\delta_3^{(\mu)}$ with the real axis and Q the intersection of $\delta_1^{(\mu)}\delta^{(\mu)}$ with the imaginary axis. By Riemann mapping theorem, there exists a conformal mapping $\Psi_\alpha^{(\mu)}$ from $OP\delta^{(\mu)}Q$ onto the first quadrant of the unit disk, with O, P, Q mapped onto $0, 1, i$, respectively. It maps $\delta^{(\mu)}$ onto a point $\hat{\delta} = e^{i\theta}$ on the unit circle, $0 < \theta < \pi/2$. By the reflection principle, we can extend it to a conformal mapping $\Psi_\alpha^{(\mu)}$ from $R_\alpha^{(\mu)}$ onto the unit disk, sending $\delta^{(\mu)}, \delta_1^{(\mu)}, \delta_2^{(\mu)}, \delta_3^{(\mu)}$ onto the four points $\hat{\delta} = e^{i\theta}, \hat{\delta}_1 = -e^{-i\theta}, \hat{\delta}_2 = -e^{i\theta}, \hat{\delta}_3 = e^{-i\theta}$, respectively. Note that if $\mu = n < \infty$, then $\Psi_\alpha^{(\mu)}$ is locally $2n$ to 1 at each of the points $\delta^{(\mu)}, \delta_1^{(\mu)}, \delta_2^{(\mu)}, \delta_3^{(\mu)}$ since the angles of $R_\alpha^{(\mu)}$ at these points are $\pi/2n$. Next we map the unit disk conformally onto the lower half-plane, with the unit circle onto the real axis and $\hat{\delta}, \hat{\delta}_1, \hat{\delta}_2, \hat{\delta}_3$ onto the points ∞, e_1, e_2, e_3 , respectively, such that $e_1 > e_2 > e_3$ and $e_1 + e_2 + e_3 = 0$. Then by the Schwarz-Christoffel mapping, the lower half-plane is mapped conformally onto $I_{\hat{\tau}}$ for some $\hat{\tau}$ with $\hat{\tau} = i|\hat{\tau}|$, $|\hat{\tau}| > 0$, and ∞, e_1, e_2, e_3 onto $0, 1, 1 + \hat{\tau}, \hat{\tau}$, respectively. Finally, $I_{\hat{\tau}}$ can be mapped conformally onto I_τ for some τ with $\tau = i|\tau|$, $|\tau| > 0$ such that $0, 1, 1 + \hat{\tau}, \hat{\tau}$ are mapped onto $1 + \tau, \tau, 0, 1$, respectively. The composition of all these is exactly the required conformal mapping $\phi_\alpha^{(\mu)}$.

(b) Suppose there exist $\tilde{\tau} = i|\tilde{\tau}|$, $|\tilde{\tau}| > 0$ and a conformal mapping $\tilde{\phi}_\alpha^{(\mu)}$ from $R_\alpha^{(\mu)}$ onto $I_{\tilde{\tau}}$, such that $\tilde{\phi}_\alpha^{(\mu)}(\delta^{(\mu)}) = 1 + \tilde{\tau}$, $\tilde{\phi}_\alpha^{(\mu)}(\delta_1^{(\mu)}) = \tilde{\tau}$, $\tilde{\phi}_\alpha^{(\mu)}(\delta_2^{(\mu)}) = 0$, $\tilde{\phi}_\alpha^{(\mu)}(\delta_3^{(\mu)}) = 1$. Then $g = (\tilde{\phi}_\alpha^{(\mu)}) \circ (\phi_\alpha^{(\mu)})^{-1}$ is a conformal mapping from I_τ onto $I_{\tilde{\tau}}$ such that $0, 1, 1 + \tau, \tau$ are mapped onto $0, 1, 1 + \tilde{\tau}, \tilde{\tau}$, respectively. By the reflection principle, g can be extended to a conformal mapping from \mathbf{C} onto itself such that $g(0) = 0, g(1) = 1$. It follows that $g = \text{id}$, hence $\tau = \tilde{\tau}$ and $\phi_\alpha^{(\mu)} = \tilde{\phi}_\alpha^{(\mu)}$.

REMARK. From this uniqueness property, it follows that $\phi_\alpha^{(\mu)}(0) = \frac{1}{2}(1 + \tau)$, and $\phi_\alpha^{(\mu)}$ maps the x -axis onto the line L_1 through $\frac{1}{2}(1 + \tau)$ and parallel to the real axis, the y -axis onto the line L_2 through $\frac{1}{2}(1 + \tau)$ and parallel to the imaginary axis. Also, each quadrant goes to the corresponding quadrant, and points inverse in the x -axis are mapped onto points inverse in L_1 ; points inverse in the y -axis are mapped onto points inverse in L_2 .

(c) $\Phi^{(\mu)}$ is 1-1. Suppose $\Phi^{(\mu)}(\alpha) = \Phi^{(\mu)}(\tilde{\alpha}) = \tau$. Then $h = (\phi_\alpha^{(\mu)})^{-1} \circ (\phi_{\tilde{\alpha}}^{(\mu)})$ is a conformal mapping from $R_{\tilde{\alpha}}^{(\mu)}$ onto $R_\alpha^{(\mu)}$ with vertices going to the corresponding vertices. By the symmetric properties of the mappings, it follows that $h(0) = 0, h'(0) > 0$; it follows that $h = \text{id}$, hence $\alpha = \tilde{\alpha}$. The fact that $\Phi^{(\mu)}$ is continuous, monotonic-decreasing and $|\tau| \rightarrow \infty$ as $\alpha \rightarrow 0, |\tau| \rightarrow 0$ as $\alpha \rightarrow 1$ (hence $\Phi^{(\mu)}$ is onto) follows from the general theory of functions. (See, for example, Triebel [4, §5].)

THEOREM 3. *Given a fixed τ ($\tau = i|\tau|$, $|\tau| > 0$). Let $\alpha_\mu = (\Phi^{(\mu)})^{-1}(\tau)$. Then the Fuchsian group $G_{\alpha_\mu}^{(\mu)}$ is completely determined. Write its generators A, B , as A_μ, B_μ . Let $\psi_\mu = (\phi_{\alpha_\mu}^{(\mu)})^{-1}$. We can extend ψ_μ analytically to \mathbb{C} . We shall call the extended mapping ψ_μ again. Then*

(i) $A_\mu \circ \psi_\mu = \psi_\mu \circ \hat{A}$, $B_\mu \circ \psi_\mu = \psi_\mu \circ \hat{B}$, where $\hat{A}: z \mapsto z+1$, $\hat{B}: z \mapsto z+\tau$, $z \in \mathbb{C}$.

(ii) $A_n(w) \rightarrow A_\infty(w)$, $B_n(w) \rightarrow B_\infty(w)$ normally on Δ , as $n \rightarrow \infty$.

(iii) $\psi_n(z) \rightarrow \psi_\infty(z)$ normally on \mathbb{C} , as $n \rightarrow \infty$, (with interpretation to the corresponding branches).

Here normal convergence means uniform convergence on each compact subset.

PROOF. (a) Note that if we apply inversion to $R_{\alpha_\mu}^{(\mu)}$ in its four sides, we get four quadrilaterals adjacent to $R_{\alpha_\mu}^{(\mu)}$, with circular arcs perpendicular to the unit circle as boundaries. Repeating this process, we can fill up (without overlapping) Δ with an infinite system of quadrilaterals. On the other hand, from the construction of $A_\mu, B_\mu, R_{\alpha_\mu}^{(\mu)}$, if we apply all elements $T \in G_{\alpha_\mu}^{(\mu)}$ to $R_{\alpha_\mu}^{(\mu)}$, we get exactly the same system of quadrilaterals. For example, the quadrilateral $A_\mu(R_{\alpha_\mu}^{(\mu)})$ is exactly the one obtained by reflecting $R_{\alpha_\mu}^{(\mu)}$ in the side $\delta_3^{(\mu)}\delta_2^{(\mu)}$, and $B_\mu(R_{\alpha_\mu}^{(\mu)})$ is the one obtained by reflecting $R_{\alpha_\mu}^{(\mu)}$ in the side $\delta_1^{(\mu)}\delta_2^{(\mu)}$. From this remark and the symmetric properties of $\phi_{\alpha_\mu}^{(\mu)}$, (i) is easily verified.

Furthermore, it is well known that in $G_{\alpha_\infty}^{(\infty)}$, the set of transformations conjugate to $(B_\infty A_\infty B_\infty^{-1} A_\infty^{-1})^m$, m an integer, is exactly all the parabolic transformations in the group. Similarly, the set of all transformations in $G_{\alpha_n}^{(n)}$ conjugate to $(B_n A_n B_n^{-1} A_n^{-1})^m$, m an integer, is exactly all the elliptic transformations in the group $G_{\alpha_n}^{(n)}$. Therefore $\Lambda_\infty = \{T(\delta^{(\infty)}) \mid T \in G_{\alpha_\infty}^{(\infty)}\}$ is the set of all parabolic fixed points of the group $G_{\alpha_\infty}^{(\infty)}$ and $\Lambda_n = \{T(\delta^{(n)}) \mid T \in G_{\alpha_n}^{(n)}\}$ is the set of all elliptic fixed points of the group $G_{\alpha_n}^{(n)}$ inside Δ . Let Ω_∞ be the set of all image points of $\delta^{(\infty)}, \delta_1^{(\infty)}, \delta_2^{(\infty)}, \delta_3^{(\infty)}$ under inversions in the sides, then clearly $\Omega_\infty = \Lambda_\infty$. Define Ω_n similarly, then $\Omega_n = \Lambda_n$.

(b) Let H be the group generated by \hat{A}, \hat{B} , then it is well known that $S = \mathbb{C}/H$ is a torus. Let $f: \mathbb{C} \rightarrow S$ be the natural projection.

Case (i). $\mu = n < \infty$. Recall that $\Psi_{\alpha_n}^{(n)}$ is locally $2n$ to 1 at each of the points $\delta^{(n)}, \delta_1^{(n)}, \delta_2^{(n)}, \delta_3^{(n)}$. Hence $\phi_{\alpha_n}^{(n)}$ is locally n to 1 at each of the points $\delta^{(n)}, \delta_1^{(n)}, \delta_2^{(n)}, \delta_3^{(n)}$. (Note that in the present case the inverse of the Schwarz-Christoffel mapping is 2 to 1.) Then extend $\phi_{\alpha_n}^{(n)}$ to Δ by reflection. From part (i), the composite mapping $\pi_{\alpha_n}^{(n)} = f \circ \phi_{\alpha_n}^{(n)}$ can be regarded as a covering projection from Δ onto S and $\Delta/G_{\alpha_n}^{(n)}$ is conformally equivalent to S . Furthermore, $\pi_{\alpha_n}^{(n)}$ is locally n to 1 at each point of Λ_n . Clearly $\pi_{\alpha_n}^{(n)}$ maps Λ_n onto one single point $P \in S$. In conclusion, $\pi_{\alpha_n}^{(n)}$ can be regarded as a (ramified) covering projection

from Δ onto the Riemann surface with signature (S, P, n) as specified by the Koebe theorem. (See §II.) And $G_{\alpha_n}^{(n)}$ is exactly a representing Fuchsian group of (S, P, n) .

Case (ii). $\mu = \infty$. Then extend $\phi_{\alpha_\infty}^{(\infty)}$ analytically to $\hat{\Delta} = \Delta \cup \Lambda_\infty$. From (i) again, the composite mapping $\pi_{\alpha_\infty}^{(\infty)} = f \circ \phi_{\alpha_\infty}^{(\infty)}$ can be regarded as a covering projection from $\hat{\Delta}$ onto S and $\hat{\Delta}/G_{\alpha_\infty}^{(\infty)}$ is conformally equivalent to S . $\pi_{\alpha_\infty}^{(\infty)}$ maps Λ_∞ onto one point $P \in S$ also. And $\Delta/G_{\alpha_\infty}^{(\infty)}$ is conformally equivalent to $S - \{P\}$. It follows that we can regard $\pi_{\alpha_\infty}^{(\infty)}$ as a covering projection onto the Riemann surface with signature (S, P, ∞) and $G_{\alpha_\infty}^{(\infty)}$ is exactly a representing Fuchsian group as specified in Koebe's theorem. Therefore, by the continuity theorem for Fuchsian groups (see §II), assertion (ii) follows immediately.

(c) Consider the branch ψ_n , which maps I_τ onto $R_{\alpha_n}^{(n)}$. Since $R_{\alpha_n}^{(n)} \subset \Delta$, $\{\psi_n\}$ is a normal family of analytic functions, i.e., each infinite sequence of members of this family has a normally convergent subsequence. We shall prove that the limit function of such a subsequence is always ψ_∞ , then clearly $\psi_n \rightarrow \psi_\infty$ normally, as $n \rightarrow \infty$. We may assume that $\psi_n \rightarrow \psi$ normally as $n \rightarrow \infty$, where ψ is defined and analytic in I_τ . We shall prove that $\psi = \psi_\infty$. Recall that $\delta^{(n)}$ is the fixed point of $D_n = B_n A_n B_n^{-1} A_n^{-1}$ inside Δ , and $\delta^{(\infty)}$ is the fixed point of $D_\infty = B_\infty A_\infty B_\infty^{-1} A_\infty^{-1}$. From (ii), it follows that $D_n \rightarrow D_\infty$ normally, as $n \rightarrow \infty$. Hence $\delta^{(n)} \rightarrow \delta^{(\infty)}$ as $n \rightarrow \infty$. Therefore, $\psi(1+\tau) = \psi_\infty(1+\tau) = \delta^{(\infty)}$; similarly, $\psi(0) = \psi_\infty(0)$, $\psi(1) = \psi_\infty(1)$, $\psi(\tau) = \psi_\infty(\tau)$. It follows from (i) and (ii) that $A_\infty(\psi(z)) = \psi(z+1)$, $B_\infty(\psi(z)) = \psi(z+\tau)$. Next we extend ψ analytically to \mathbb{C} by these two relations. Then $\mathbb{C}_\tau = \mathbb{C} - \{k+l\tau \mid k, l \text{ integers}\}$ is mapped onto Δ and all the lattice points $(k+l\tau, k, l \text{ integers})$ onto the unit circle. Therefore $\zeta = \psi \circ \psi_\infty^{-1}$ is a conformal mapping from Δ onto itself and has fixed points $\delta^{(\infty)}$, $\delta_1^{(\infty)}$, $\delta_2^{(\infty)}$, $\delta_3^{(\infty)}$. Therefore $\zeta = \text{id}$.

THEOREM 4. Let τ ($\tau = i|\tau|$, $|\tau| > 0$) be given.

(i) For any integer $n \geq 2$, there exists a unique real number $C = C^{(n)}$ such that ψ_n is a solution to (3).

(ii) There exists a unique real number K such that ψ_∞ is a solution to (4).

(iii) $K = \lim_{n \rightarrow \infty} C^{(n)}$.

PROOF. (a) By Theorem 3(i), if we analytically continue a branch of ψ_n along a closed loop in \mathbb{C}_τ , we return to a function element which is the previous one followed by an element of the group $G_{\alpha_n}^{(n)}$. Hence $\{\psi_n, z\}$ is single-valued and analytic in \mathbb{C}_τ . It is also clear that $\{\psi_n, z\}$ is a doubly periodic function with periods 1, τ .

In order to study the singularity of $\{\psi_n, z\}$ at a lattice point z , we analytically continue ψ_n along a closed loop L enclosing z once and in

the positive direction. We shall return to a function element which is the previous one followed by an elliptic transformation of order n . Therefore in a neighborhood of \hat{z} , $\psi_n(z) = (z - \hat{z})^{1/n} \Phi(z)$, where $\Phi(z)$ is analytic in a neighborhood of \hat{z} and $\Phi(\hat{z}) \neq 0$. (See Ford [2, Chapter XI].) Direct computation shows that $\{\psi_n, z\}$ has a pole of order 2 at \hat{z} with $\frac{1}{2}(1 - 1/n^2)(z - \hat{z})^{-2}$ as the leading term in the Laurent series expansion about \hat{z} . This result holds for any lattice point. From the theory of doubly periodic functions, we conclude that $\{\psi_n, z\} = \frac{1}{2}(1 - 1/n^2)\wp(z; 1, \tau) + 2C^{(n)}$, where $C^{(n)}$ is a real constant uniquely determined by ψ_n .

(b) Similarly, $\{\psi_\infty, z\}$ is a single-valued doubly periodic function analytic in C_τ , with periods 1, τ . If we continue ψ_∞ analytically along the loop L as specified above, we shall return to a function element which is the previous one followed by a parabolic transformation. Therefore in a neighborhood of the lattice point \hat{z} , $\psi_\infty(z) = T(\log \Psi(z))$, where T is a linear transformation, $\Psi(z)$ is analytic in a neighborhood of \hat{z} with $\Psi(\hat{z}) = 0$, $\Psi'(\hat{z}) = 1$. (See Ford [2, Chapter XI].) Computation shows that $\{\psi_\infty, z\}$ has a pole of order 2 at \hat{z} with $\frac{1}{2}(z - \hat{z})^{-2}$ as the leading term in the Laurent series expansion about \hat{z} . Therefore, $\{\psi_\infty, z\} = \frac{1}{2}\wp(z; 1, \tau) + 2K$, where K is a real constant uniquely determined by ψ_∞ .

(c) Proof of part (iii) is clear.

THEOREM 5. Let τ ($\tau = i|\tau|$, $|\tau| > 0$) be given.

(i) For any integer $n \geq 2$, there exists a unique real number $C = C^{(n)}$ such that (1) possesses two linearly independent solutions.

(ii) Suppose $\eta_1^{(n)}(z)$, $\eta_2^{(n)}(z)$ are any two linearly independent solutions of (1) with $C = C^{(n)}$, then the limits $\eta_1^{(\infty)} = \lim_{n \rightarrow \infty} \eta_1^{(n)}$, $\eta_2^{(\infty)} = \lim_{n \rightarrow \infty} \eta_2^{(n)}$ exist, and form two linearly independent solutions of (2) with $K = \lim_{n \rightarrow \infty} C^{(n)}$.

PROOF. Immediate from Theorems 1 and 4.

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