PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 28, Number 2, May 1971

ON THE CONVERGENCE OF MULTIPLICATIVELY ORTHOGONAL SERIES

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ABSTRACT. G. Alexits and A. Sharma have recently shown that if $\{\varphi_n\}_{n=1}^{\infty}$ is a uniformly bounded multiplicatively orthogonal system on a finite measure space and if $\{c_n\}_{n=1}^{\infty}$ is a sequence of real numbers with $\sum_{n=1}^{\infty} c_n^2 < \infty$, then the partial sums $\sum_{k=1}^{n} c_k\varphi_k$ converge almost everywhere. We give here a simple proof of this result.

Let $(X, \mathcal{B}, \mu)$ be a measure space, with μ a finite nonnegative measure, and let $f_n: X \to \boldsymbol{R}$, $n = 1, 2, \cdots$, be an orthonormal system on $(X, \mathcal{B}, \mu)$, (i.e. $f_n \in L^2(X, \mathcal{B}, \mu)$ with $\int_X f_n f_m \, d\mu = \delta_{m,n}$). Let $c_n \in R$, $n = 1, 2, \cdots$, and define s_n by $s_n(x) = \sum_{\nu=1}^{n} c_\nu f_\nu(x)$. Then the classical result of Menchoff states that s_n converges a.e. as $n \to \infty$, provided $\sum_{n=1}^{\infty} c_n^2 (\log n)^2 < \infty$. Menchoff also showed that for a general orthonormal system this is the best result possible. For particular orthonormal systems we can get better results; for example, if $X = T$, and $\mu =$ Lebesgue measure on T, and $f_n(x) = \cos nx$, or $f_n(x) = \sin nx$, then it follows from the famous result of Carleson that s_n converges a.e. as $n \to \infty$ provided $\sum_{n=1}^{\infty} c_n^2 < \infty$. In a preprint of a paper to appear in Acta. Math. Acad. Sci. Hungar., G. Alexits and A. Sharma prove a similar result for uniformly bounded multiplicatively orthogonal systems. (We say $\{\varphi_n\}_{n=1}^{\infty}$ is a uniformly bounded multiplicatively orthogonal system on $(X, \mathcal{B}, \mu)$ if $\varphi_n \in L^\infty(X, \mathcal{B}, \mu)$ with $\|\varphi_n\|_\infty \leqq M$ for some M and all n, and if given any $m = 1, 2, \cdots$, and $1 \leqq \nu_1 < \nu_2 < \cdots < \nu_m$, then $\int_X \varphi_{\nu_1} \cdots \varphi_{\nu_m} \, d\mu = 0$.)

Alexits and Sharma prove the following:

THEOREM. *Let $\{\varphi_n\}_{n=1}^{\infty}$ be a uniformly bounded multiplicatively orthogonal system on $(X, \mathcal{B}, \mu)$. Let $c_n \in \boldsymbol{R}$, $n = 1, 2, \cdots$, and let $s_n(x) = \sum_{\nu=1}^{n} c_\nu \varphi_\nu(x)$. Then s_n converges a.e. as $n \to \infty$ provided $\sum_{n=1}^{\infty} c_n^2 < \infty$.*

The proof of this theorem by Alexits and Sharma involves some difficult constructions; we give here a short and simple proof.

We may suppose without loss of generality that $|\varphi_n(x)| \leqq 1$ for all $x \in X$ and for all n. Let $\{\psi_n\}_{n=0}^{\infty}$ be the product system associated with $\{\varphi_n\}_{n=1}^{\infty}$; i.e.

Received by the editors July 27, 1970.

AMS 1970 subject classifications. Primary 40A05, 42A60, 42A20.

Key words and phrases. Uniformly bounded multiplicatively orthogonal systems, almost everywhere convergence.

$$\psi_n = \varphi_{\nu_1+1} \cdots \varphi_{\nu_m+1} \quad \text{for } n = 2^{\nu_1} + \cdots + 2^{\nu_m},$$
$$\psi_0 \equiv 1.$$

Note the following two facts:

$$\int_X \psi_n \, d\mu = 0, \qquad n = 1, 2, \cdots; \tag{1}$$

$$\sum_{k=0}^{2^m-1} \psi_k(x)\psi_k(y) = \prod_{k=1}^{m} (1 + \varphi_k(x)\varphi_k(y)) \geqq 0 \quad \text{for all } x, y \in X. \tag{2}$$

Define $n(x)$ to be the least index such that $s_{n(x)}(x) = \max_{1 \leq \nu \leq n} s_\nu(x)$. We have

$$s_n(x) = \sum_{k=0}^{2^n-1} a_k\psi_k(x),$$

where

$$\begin{aligned} a_k &= c_{\nu+1} \quad \text{if } k = 2^\nu, \\ &= 0 \qquad \text{otherwise,} \end{aligned}$$

and so $s_{n(x)}(x) = \sum_{k=0}^{2^{n(x)}-1} a_k\psi_k(x)$.

Let $(Y, \mathfrak{A}, \omega)$ be any finite measure space, and let $\{g_n\}_{n=0}^{\infty}$ be any orthonormal system on $(Y, \mathfrak{A}, \omega)$. Then

$$s_{n(x)}(x) = \int_Y \sum_{k=0}^{2^n-1} a_k g_k(t) \sum_{j=0}^{2^{n(x)}-1} \psi_j(x) g_j(t) \, d\omega(t).$$

Therefore

$$\begin{aligned} \left| \int_X s_{n(x)}(x) \, d\mu(x) \right| &= \left| \int_Y \sum_{k=0}^{2^n-1} a_k g_k(t) \int_X \sum_{j=0}^{2^{n(x)}-1} \psi_j(x) g_j(t) \, d\mu(x) \, d\omega(t) \right| \\ &\leqq \left\{ \int_Y \left[\sum_{k=0}^{2^n-1} a_k g_k(t) \right]^2 d\omega(t) \right. \\ &\qquad \left. \cdot \int_Y \left[\int_X \sum_{k=0}^{2^{n(x)}-1} \psi_k(x) g_k(t) \, d\mu(x) \right]^2 d\omega(t) \right\}^{1/2} \\ &= \left\{ \left(\sum_{k=0}^{2^n-1} a_k^2 \right) \int_Y \int_X \int_X \sum_{k=0}^{2^{n(x)}-1} \psi_k(x) g_k(t) \right. \\ &\qquad \left. \cdot \sum_{j=0}^{2^{n(y)}-1} \psi_j(y) g_j(t) \, d\mu(x) \, d\mu(y) \, d\omega(t) \right\}^{1/2}. \end{aligned}$$

Thus

$$
\begin{aligned}
&\left| \int_X s_{n(x)}(x)\, d\mu(x) \right|^2 \\
&\leqq \left(\sum_{k=1}^{n} c_k^2 \right) \int_X \int_X \int_Y \sum_{k=0}^{2^{n(x)}-1} \psi_k(x) g_k(t) \sum_{j=0}^{2^{n(y)}-1} \psi_j(y) g_j(t)\, d\omega(t)\, d\mu(x)\, d\mu(y) \\
&= \left(\sum_{k=1}^{n} c_k^2 \right) \int_X \int_X \sum_{k=0}^{2^{n(x,y)}-1} \psi_k(x)\psi_k(y)\, d\mu(x)\, d\mu(y), \\
&\qquad\qquad\qquad\qquad \text{where } n(x, y) = \min\{n(x), n(y)\}, \\
&\leqq 2\left(\sum_{k=1}^{n} c_k^2 \right) \int_X \int_X \left| \sum_{k=0}^{2^{n(y)}-1} \psi_k(x)\psi_k(y) \right| d\mu(x)\, d\mu(y) \\
&= 2\left(\sum_{k=1}^{n} c_k^2 \right) \int_X \int_X \sum_{k=0}^{2^{n(y)}-1} \psi_k(x)\psi_k(y)\, d\mu(x)\, d\mu(y) \quad \text{(using (2))} \\
&= 2\left(\sum_{k=1}^{n} c_k^2 \right) \int_X \int_X \psi_0(x)\psi_0(y)\, d\mu(x)\, d\mu(y) \qquad \text{(using (1))} \\
&= 2\left(\sum_{k=1}^{n} c_k^2 \right) [\mu(X)]^2.
\end{aligned}
$$

Hence we have

$$
\left| \int_X s_{n(x)}(x)\, d\mu(x) \right|^2 \leqq 2\left(\sum_{k=1}^{\infty} c_k^2 \right) [\mu(X)]^2.
$$

It is well known that such an estimate is sufficient in order to prove the theorem.

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