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A CLASS OF HYPO-DIRICHLET ALGEBRAS

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ABSTRACT. A method is given of constructing a new class of hypo-Dirichlet algebras of given real codimension.

1. **Introduction.** Let X be a compact Hausdorff space and A a uniform algebra on X, i.e., a uniformly closed subalgebra of $C(X)$, the space of continuous functions on X, that contains the constants and separates points on X. Denote the real parts of the functions in A by $\operatorname{Re} A$, the set of invertible elements of A by A^{-1}, the set of logarithms of moduli of functions in A by $\log A$. Let $C(X)$ denote the space of real continuous functions on X. A uniform algebra on X is called a *hypo-Dirichlet* algebra if, in addition, there exist $f_1, \cdots, f_n$ in A^{-1}, such that the (real) vector space spanned by $\operatorname{Re} A$ and $\log |f_1|, \cdots, \log |f_n|$ is dense in $C(X)$. The minimal number of such 'f_i' required shall be called the *codimension* of $\operatorname{Re} A$. Hypo-Dirichlet algebras were first studied by Wermer [6], and further investigated by Ahern and Sarason [1]. The object here is to exhibit a class of examples of such algebras. The proofs of several of the lemmas in this paper are modeled after [2].

2. **The algebra** A. Let Γ be the annulus $\{Z:1\leqq|Z|\leqq 2\}$, $\gamma_1 = \{Z:|Z|=1\}$ and $\gamma_2=\{Z:|Z|=2\}$. Let Ψ be a homeomorphism of γ_1 on γ_2 which is orientation-preserving and singular, i.e., maps a Borel set of one-dimensional Lebesgue measure 0 onto a set of measure 4π. Let $B=\{f\in C(\Gamma):f$ is analytic in $\operatorname{int}(\Gamma)\}$, and $A = \{f\in B:f(Z)=f(\Psi(Z))\}$ for all $Z\in\gamma_1$. Let $A_\Psi=A$ restricted to γ_1. Then A_Ψ is a uniformly closed algebra of continuous functions on γ_1, which contains the constants.

THEOREM. *A_Ψ is a hypo-Dirichlet algebra on γ_1, and* $\operatorname{Re} A_\Psi$ *has codimension* 1 *in* $C_R(\gamma_1)$.

DEFINITION 1. A (complex Borel) measure ν on $\gamma_1\cup\gamma_2$ is odd if for each Borel set $E\subset\gamma_1$, $\nu(E)=-\nu(\Psi(E))$.

DEFINITION 2. H denotes the class of measures of the form: $g(Z)\,dZ$ on $\gamma_1\cup\gamma_2$, where g is any function in the L^1 closure of B restricted to $\gamma_1\cup\gamma_2$.

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DEFINITION 3. W is the space of measures $\mu+\nu$ with $\mu\in H$, ν odd. $\overline{W}$ is the weak * closure of W in the space of measures on $\gamma_1\cup\gamma_2$.

DEFINITION 4. A measure λ on $\gamma_1\cup\gamma_2$ *annihilates* A if $\int f\,d\lambda=0$, all $f\in A$.

Clearly, every measure in W annihilates A. Also, if λ annihilates A, then $\lambda\in\overline{W}$.

NOTE. The measure $-i\cdot(dZ/Z)=d\theta$ is a real measure which annihilates B, and it is readily seen that the only real annihilators of B are of the form: $\alpha\cdot d\theta$, α a real constant.

LEMMA 1. *If $\mu\in H$, ν odd, then $\|\nu\|\leqq 16\|\mu+\nu\|$.*

PROOF. Let E be any Borel subset of γ_1 and let m represent Lebesgue measure. Then there are disjoint sets F and G with $E=F\cup G$, $m(F)=m(\Psi(G))=0$. Let $K=\|\mu+\nu\|$, then $|\nu(F)|=|\nu(F)+\mu(F)|\leqq K$, since μ is absolutely continuous. $|\nu(G)|=|\nu(\Psi(G))|=|(\mu+\nu)(\Psi(G))|\leqq K$ for the same reason. Hence $\|\nu\|\leqq 16\,K$. q.e.d.

LEMMA 2. *Then $W=\overline{W}$.*

PROOF. $Q=\{\mu+\nu:\mu\in H,\ \nu \text{ odd},\ \|\mu\|\leqq 1,\ \|\nu\|\leqq 1\}$ is compact. The Krein-Smulian theorem [4, p. 429] then implies $W=\overline{W}$. q.e.d.

LEMMA 3. *If ν is an odd measure, then ν is absolutely continuous with respect to arc length on $\gamma_1\cup\gamma_2$ iff $\nu=0$.*

PROOF. Suppose ν is absolutely continuous. Let E be a Borel subset of γ_1. Then there are disjoint sets F and G with $E=F\cup G$, $m(F)=m(\Psi(G))=0$. Hence $\nu(F)=0$, since ν is absolutely continuous $\nu(G)=-\nu(\Psi(G))=0$ for the same reason. Hence $\nu(E)=0$. q.e.d.

LEMMA 4. *Every real annihilator, λ, of A has the form: $\lambda=\nu+\alpha\cdot d\theta$, where ν is odd and α is a real scalar.*

PROOF. Since W is weak * closed, we conclude that if λ is a measure on $\gamma_1\cup\gamma_2$, which annihilates A, then $\lambda=\mu+\nu$, $\mu\in H$, ν odd. Write $\mu=\mu_1+i\mu_2$, $\nu=\nu_1+i\nu_2$ with μ_1, μ_2, ν_1, and ν_2 real. If λ is real $\mu_2+\nu_2=0$. Hence ν_2 is absolutely continuous, hence 0. Then $\mu=\mu_1$ and $\lambda=\mu_1+\nu_1$. q.e.d.

It follows, in particular, that Re A has codimension $\leqq 1$ in $C_R(\gamma_1)$.

LEMMA 5. *A separates the points of γ_1. Further, given Z_1, Z_2 with $1\leqq|Z_1|\leqq 2$, $1\leqq|Z_2|<2$ and $Z_1\neq Z_2$, then there exists an f in A such that $f(Z_1)\neq f(Z_2)$.*

PROOF. Let τ_1, τ_2 be two points of γ_1 and let δ_{τ_1}, δ_{τ_2} be the point masses at τ_1, τ_2 respectively. Unless A separates τ_1 and τ_2, $\delta_{\tau_1}-\delta_{\tau_2}$ will be a real annihilating measure which is not in W. Now suppose Z_1, Z_2 are interior to the annulus and A fails to separate them. Let σ_1, σ_2 be the harmonic measures for Z_1, Z_2 respectively. Then $\sigma_1-\sigma_2$ would be a real annihilating measure. Hence, $\sigma_1-\sigma_2=\nu+\alpha\cdot d\theta$, ν odd. However $\sigma_1-\sigma_2$ is absolutely continuous, therefore $\sigma_1-\sigma_2=\alpha\cdot d\theta$, contradiction. Finally, if $Z_1\in\gamma_1$ and Z_2 is interior, a similar argument applies. q.e.d.

Let T be the space obtained from the closed annulus $1\leqq|Z|\leqq 2$ by by identifying Z and $\Psi(Z)$ if $Z\in\gamma_1$. Then functions in A may be regarded as continuous functions on T. Evidently T is topologically a torus since Ψ is orientation-preserving.

LEMMA 6. *The space of maximal ideals of A (A_Ψ) is homeomorphic to T.*

PROOF. It must be shown that, if h is a homomorphism of A onto the complex numbers, then h is evaluation at some point of T. If h is not evaluation at any point of T, then for each Z, $1\leqq|Z|\leqq 2$, there is an $f_Z\in A$, with $h(f_Z)=0$, $f_Z(Z)\neq 0$. Since T is compact, we can select a finite number of functions $f_1,\cdots,f_n$ in A such that $h(f_i)=0$ and open sets Δ_i in $1\leqq|Z|\leqq 2$ such that $\bigcup_1^n\Delta_i=\{Z:1\leqq|Z|\leqq 2\}$ and $f_i\neq 0$ in Δ_i. Let σ be a representing measure for h on $\gamma_1\cup\gamma_2$, i.e., $h(f)=\int f\,d\sigma$, all $f\in A$. Then $\int f\cdot f_i\,d\sigma=h(f\cdot f_i)=h(f)\cdot h(f_i)=0$, $i=1,\cdots,n$, $f\in A$. Thus $f_i\cdot d\sigma$ annihilates A, therefore $f_i\cdot d\sigma=d\mu_i+d\nu_i$, $\mu_i\in H$, ν_i odd. Hence, $f_j(d\mu_i+d\nu_i)=f_j\cdot f_i\,d\sigma=f_i\cdot(d\mu_j+d\nu_j)$ and so $f_j\cdot d\mu_i-f_i d\mu_j=f_i\cdot d\nu_j-f_j\cdot d\nu_i$. Since the right side is odd and the left side is absolutely continuous both sides vanish. Let Φ_i denote the function in H such that $d\mu_i=\Phi_i\cdot dZ$. Then $f_j\cdot\Phi_i=f_i\cdot\Phi_j$ a.e. on $\gamma_1\cup\gamma_2$ and so $f_j\cdot\Phi_i=f_i\cdot\Phi_j$ also for $1<|Z|<2$. We can therefore unambiguously define Φ on $1\leqq|Z|\leqq 2$ by $\Phi(z)=\Phi_i(Z)\cdot(f_i(Z))^{-1}$ for $Z\in\Delta_i$. Then $\Phi\in H$.

We define a measure ν on $\gamma_1\cup\gamma_2$ by $d\nu=(f_i)^{-1}\cdot d\nu_i$ on $(\gamma_1\cup\gamma_2)\cap\Delta_i$. Then ν is well defined and odd. Then $f_i\,d\sigma=f_i\cdot\Phi\cdot dZ+f_i\,d\nu$. Since $f_i\neq 0$ on Δ_i, we deduce $d\sigma=\Phi dZ+d\nu$. But then $1=\int d\sigma=\int\Phi\cdot dZ+\int d\nu=0$. Contradiction. q.e.d.

LEMMA 7. *There is an $f\in A^{-1}$ whose logarithm is not single valued on Γ.*

PROOF. We regard A as an algebra of continuous functions on T. The circle: $|Z|=3/2$ gives rise to a one-cycle l_1 on T. Let l_2 be another one-cycle on T so that l_1 and l_2 generate $H_1(T,Z)$. By a theorem of Arens-Royden, [5], the quotient group $A^{-1}/\exp(A)$ is isomorphic to $H^1(T,Z)$. If T is the torus, $H^1(T,Z)$ is a free abelian group on two generators. Let g_1, g_2 be two elements of A^{-1} representing these generators. Write $g_1 = e^{h_1}$, $g_2 = e^{h_2}$, where h_1, h_2 are (multi-valued) analytic functions on Γ. Let h_1 have period $2n\pi i$ on l_2, h_2 have period $2m\pi i$ on l_2. Then $m \cdot h_1 - n \cdot h_2$ has 0 period on l_2. Suppose it also had 0 period on l_1. Then $g_1^m \cdot g^{-n} = e^h$ for some $h \in A$. This contradicts the choice of g_1, g_2. Hence $mh - nh$ has period $\neq 0$ on l. Therefore $f = g_1^m \cdot g_2^{-n}$ is the desired element of A^{-1}. q.e.d.

PROOF OF THEOREM. We must show that there is an $f \in A_\psi^{-1}$ such that $\log |f| \notin$ closure Re A. We claim the f of Lemma 7 is such a function. Define a linear functional L on $C_R(\gamma_1 \cup \gamma_2)$ by $L(U) = (1/2\pi)\int_{|Z|=3/2} dv$ where v is the harmonic conjugate of U. Then L is continuous and linear. $L(g) = 0$ for $g \in \operatorname{Re} A^{-1}$, but $L(\log |f|) \neq 0$ since $\int_{|Z|=3/2} (\arg f) \neq 0$. Therefore $\log |f| \notin$ closure Re A^{-1}. q.e.d.

NOTE. By identifying n circles instead of 2, in a similar manner, we can construct a hypo-Dirichlet algebra that has real part of codimension $n-1$.

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