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ON MATRICES WHOSE NONTRIVIAL REAL LINEAR COMBINATIONS ARE NONSINGULAR

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ABSTRACT. Let F be the real field R, the complex field C, or the skew field H of quaternions, and $d(F)$ the real dimension of F. We shall write $F(n)$ (resp. $F_x(n)$) for the maximum number of $n\times n$ matrices (resp. $n\times n$ matrices with property x) with elements in F whose nontrivial linear combinations with real coefficients are nonsingular and x will stand for hermitian (h), skew-hermitian (sk-h), symmetric (s), or skew-symmetric (sk-s). If n is a positive integer, we write $n=(2a+1)2^b$, where $b=c+4d$ and a, b, c, d are nonnegative integers with $0\leqq c<4$, and define the Hurwitz-Radon function ρ of n as $\rho(n)=2^c+8d$. It is known [1], [2] that

$$R(n) = \rho(n),\ C(n) = 2b + 2,\ H(n) = \rho(\tfrac{1}{2}n) + 4,$$
$$F_h(n) = \mathrm{F}(\tfrac{1}{2}n) + 1, \text{ for } F = R, C \text{ or } H,$$

where $\rho(\frac{1}{2}n)=F(\frac{1}{2}n)=0$ if n is odd. In this note we use these known results to prove the following theorems.

THEOREM 1. *$F_{\text{sk-h}}(n)=F(n)-1$, for $F=R$, C or H.*

THEOREM 2. *$F_s(n)=\rho(\frac{1}{2}n)+d(F)$, for $F=R$, C or H.*

THEOREM 3. *$F_{\text{sk-s}}(n)=\rho(2^{d(F)-1}n)-d(F)$, if $F=R$ or C or $F=H$ and $n>2$ and $H_{\text{sk-s}}(1)=0$, $H_{\text{sk-s}}(2)=4$.*

COROLLARY. *$H_s(n)=H(n)$ for all n; $H_{\text{sk-s}}(n)=H(n)$ for all $n>2$.*

From the above results, it may be interesting to note that $F_{\mathrm{h}}(n)$ and $F_{\text{sk-h}}(n)$ can be expressed by the function $F(m)$, while $F_s(n)$ and $F_{\text{sk-s}}(n)$ can be expressed by ρ and $d(F)$ except for two exceptional cases ($H_{\text{sk-s}}(1)$ and $H_{\text{sk-s}}(2)$). The corollary follows immediately from Theorems 2 and 3, the expression of $H(n)$, and the fact that $\rho(8n)=\rho(\frac{1}{2}n)+8$.

We denote by $M(n, F)$ (resp. $M(n, F_x)$) the set of all $n\times n$ matrices (resp. $n\times n$ matrices with property x) with elements in F. If $X\in M(n, F)$, we denote by X^t and X^c the transpose and conjugate of X respectively. We shall use $\{1, e_1, e_2, e_3\}$ to denote the basis of H

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(over R) and identify e_1 with the complex number $\sqrt{(-1)}$. For simplicity we shall say that $X_1, \cdots, X_p$ in $M(n, F)$ are *independent* if the nontrivial linear combinations of these matrices with real coefficients are nonsingular.

We first prove several lemmas.

LEMMA 1. *Let $K=A_0+e_1A_1\in M(n, C)$, where $A_0, A_1\in M(n, R)$. Then K is singular if and only if*

$$\begin{pmatrix} A_0 & -A_1 \\ A_1 & A_0 \end{pmatrix} \qquad (\in M(2n, R))$$

is singular.

LEMMA 2. *Let $Q=K_1+e_2K_2\in M(n, H)$, where $K_1, K_2\in M(n, C)$. Then Q is singular if and only if*

$$\begin{pmatrix} K_1 & -K_2^c \\ K_2 & K_1^c \end{pmatrix} \qquad (\in M(2n, C))$$

is singular.

LEMMA 3. *Let $Q=A_0+e_1A_1+e_2A_2+e_3A_3\in M(n, H)$, where $A_0, A_1, A_2, A_3\in M(n, R)$. Then Q is singular if and only if the matrix*

$$\begin{pmatrix} A_0 & -A_2 & -A_1 & A_3 \\ A_2 & A_0 & A_3 & A_1 \\ A_1 & -A_3 & A_0 & -A_2 \\ -A_3 & -A_1 & A_2 & A_0 \end{pmatrix} \qquad (\in M(4n, R))$$

is singular.

LEMMA 4. *$F_{\text{sk-h}}(n)+1\leqq F(n)$, for $F=R$, C or H.*

Lemmas 1 and 2 follow from comparison of the components of the equation $Xv=0$, where $v\ (\neq 0)\in F^n$ and $X\in M(n, F)$, and Lemma 3 is an immediate consequence of Lemmas 1 and 2. Lemma 4 follows from the fact that if $X\in M(n, F_{\text{sk-h}})$, then X has no nonzero real eigenvalue.

LEMMA 5. $H_{\text{h}}(n)+2\leqq R_{\text{sk-h}}(4n)$.

PROOF. Let $Q=A_0+e_1A_1+e_2A_2+e_3A_3\in M(n, H_{\text{h}})$ be nonsingular, where $A_i\ (i=0, \cdots, 3)\in M(n, R)$. Then $A_0=A_0^t$, $A_i=-A_i^t$ for $i=1, 2, 3$ and by Lemma 3, the matrix

$$\tilde{Q} = \begin{pmatrix} A_2 & A_0 & A_3 & -A_1 \\ -A_0 & A_2 & A_1 & A_3 \\ A_3 & A_1 & -A_2 & A_0 \\ -A_1 & A_3 & -A_0 & -A_2 \end{pmatrix}$$

is nonsingular. Now the following matrices

$$\tilde{Q}, \quad \begin{pmatrix} 0 & 0 & I & 0 \\ 0 & 0 & 0 & -I \\ -I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 & I \\ 0 & 0 & I & 0 \\ 0 & -I & 0 & 0 \\ -I & 0 & 0 & 0 \end{pmatrix},$$

where 0 and I are the $n\times n$ zero and identity matrices respectively, are in $M(4n, R_{\text{sk-h}})$ and they are independent. Thus Lemma 5 is proved.

LEMMA 6. $R_{\text{h}}(n)+2\leqq H_{\text{sk-h}}(n)$.

PROOF. Let $A\in M(n, R_{\text{h}})$ be nonsingular. Then e_1A is nonsingular and e_1A, e_2I, e_3I are in $H_{\text{sk-h}}(n)$ and they are independent. Thus Lemma 6 is proved.

Now we prove Theorem 1. Since $K\in M(n, C_{\text{h}})$ if and only if $e_1K\in M(n, C_{\text{sk-h}})$, we have $C_{\text{sk-h}}(n)=C_{\text{h}}(n)=C(n)-1$. From Lemmas 4 and 6 we have $R_{\text{h}}(n)+2\leqq H_{\text{sk-h}}(n)\leqq H(n)-1$ $(=R_{\text{h}}(n)+2)$. Hence $H_{\text{sk-h}}(n)=H(n)-1$. From Lemmas 4 and 5 we have $H_{\text{h}}(n)+2\leqq R_{\text{sk-h}}(4n)\leqq R(4n)-1$ $(=H_{\text{h}}(n)+2)$. Hence $R_{\text{sk-h}}(4n)=R(4n)-1$. Now, if m is odd, then $R_{\text{sk-h}}(m)=0=R(m)-1$, and $1\leqq R_{\text{sk-h}}(2m)\leqq R(2m)-1=1$. Hence $R_{\text{sk-h}}(n)=R(n)-1$ for all n and Theorem 1 is proved.

In order to prove Theorem 2 we need the following lemmas.

LEMMA 7. $R_{\text{h}}(n)+1\leqq C_{\text{s}}(n)$.

PROOF. Let $A\in M(n, R_{\text{h}})$ be nonsingular. Then A and e_1I are in $M(n, C_{\text{s}})$ and they are independent.

LEMMA 8. $C_{\text{s}}(n)+2\leqq H_{\text{s}}(n)$.

PROOF. Let $K\in M(n, C_{\text{s}})$ be nonsingular. Then K, e_2I, e_3I are in $M(n, H_{\text{s}})$ and they are independent.

We now prove Theorem 2. By Lemmas 7 and 8 we have $R_{\text{h}}(n)+1\leqq C_{\text{s}}(n)\leqq H_{\text{s}}(n)-2\leqq H(n)-2$ $(=R_{\text{h}}(n)+1)$. Hence all these inequalities are equalities. Since $R_{\text{s}}(n)=R_{\text{h}}(n)$, Theorem 2 is proved.

In order to prove Theorem 3 which is the difficult part of this note, we need the following lemmas.

LEMMA 9. $H_h(n)+1\leqq C_{sk\text{-}s}(2n)$.

PROOF. Let $Q=K_1+e_2K_2\in M(n, H_h)$ be nonsingular, where K_1, $K_2\in M(n, C)$. Then $K_1\in M(n, C_h)$ and $K_2\in M(n, C_{sk\text{-}s})$ and by Lemma 2, the matrix

$$\tilde{Q}=\begin{pmatrix} K_2 & -K_1^c \\ K_1 & K_2^c \end{pmatrix}$$

is nonsingular. Now since

$$\tilde{Q} \quad \text{and} \quad \begin{pmatrix} 0 & e_1 I \\ -e_1 I & 0 \end{pmatrix}$$

are in $M(2n, C_{sk\text{-}s})$ and they are independent, Lemma 9 is proved.

LEMMA 10. $C_{sk\text{-}s}(n)+2\leqq R(2n)$.

PROOF. Let $K=A_0+e_1A_1\in M(n, C_{sk\text{-}s})$ be nonsingular, where A_0, $A_1\in M(n, R)$. Then by Lemma 1

$$\begin{pmatrix} A_0 & -A_1 \\ A_1 & A_0 \end{pmatrix}$$

is nonsingular. Now since

$$\begin{pmatrix} A_0 & -A_1 \\ A_1 & A_0 \end{pmatrix}, \quad \begin{pmatrix} -I & 0 \\ 0 & I \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$$

are in $M(2n, R)$ and they are independent, Lemma 10 is proved.

LEMMA 11. $C_{sk\text{-}s}(n)+6\leqq H_{sk\text{-}s}(4n)$.

PROOF. Let $K\in M(n, C_{sk\text{-}s})$ be nonsingular. Then the following matrices

$$\begin{bmatrix} K & 0 & 0 & 0 \\ 0 & K & 0 & 0 \\ 0 & 0 & K & 0 \\ 0 & 0 & 0 & K \end{bmatrix}, \quad \begin{bmatrix} 0 & e_2I & 0 & 0 \\ -e_2I & 0 & 0 & 0 \\ 0 & 0 & 0 & e_2I \\ 0 & 0 & -e_2I & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & -e_3I & 0 & 0 \\ e_3I & 0 & 0 & 0 \\ 0 & 0 & 0 & e_3I \\ 0 & 0 & -e_3I & 0 \end{bmatrix},$$

$$\begin{bmatrix} 0 & 0 & e_2I & 0 \\ 0 & 0 & 0 & -e_2I \\ -e_2I & 0 & 0 & 0 \\ 0 & e_2I & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & -e_3I & 0 \\ 0 & 0 & 0 & -e_3I \\ e_3I & 0 & 0 & 0 \\ 0 & e_3I & 0 & 0 \end{bmatrix},$$

$$\begin{pmatrix} 0 & 0 & 0 & -e_2I \\ 0 & 0 & -e_2I & 0 \\ 0 & e_2I & 0 & 0 \\ e_2I & 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 & e_3I \\ 0 & 0 & -e_3I & 0 \\ 0 & e_3I & 0 & 0 \\ -e_3I & 0 & 0 & 0 \end{pmatrix}$$

are in $M(4n, H_{sk-s})$, and, by direct calculation, they are independent. Thus Lemma 11 is proved.

LEMMA 12. $H_{sk-s}(m)=4$ *if m is odd and greater than* 1.

PROOF. If m is odd, then $H_{sk-s}(m) \leqq H(m)=4$. From this it is obvious that $H_{sk-s}(m) \leqq H_{sk-s}(m+2)$. Let

$$Q = \begin{pmatrix} 0 & 1 & e_1 \\ -1 & 0 & -1 \\ -e_1 & 1 & 0 \end{pmatrix} + e_2 \begin{pmatrix} 0 & e_1 & 0 \\ -e_1 & 0 & e_1 \\ 0 & -e_1 & 0 \end{pmatrix}.$$

Then $Q \in M(3, H_{sk-s})$ and by using Lemma 2 it can be verified that Q is nonsingular. Now Q, e_1Q, e_2Q, e_3Q are in $M(3, H_{sk-s})$ and they are independent. Hence $H_{sk-s}(3) \geqq 4$ and the lemma is proved.

LEMMA 13. $H_{sk-s}(n)+1 \leqq H_{sk-s}(2n)$.

PROOF. Let $Q \in M(n, H_{sk-s})$ be nonsingular. Then

$$\begin{pmatrix} Q & 0 \\ 0 & -Q^c \end{pmatrix}$$

and

$$\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$$

are in $M(2n, H_{sk-s})$ and they are independent. Thus Lemma 13 is proved.

We now prove Theorem 3. From Lemmas 9 and 10 we have $H_h(n)+1 \leqq C_{sk-s}(2n) \leqq R(4n)-2$ $(=H_h(n)+1)$. Hence $C_{sk-s}(2n)=R(4n)-2$. If m is odd, then $C_{sk-s}(m)=0=R(2m)-2$. Hence $C_{sk-s}(n)=R(2n)-2$ for all n. From this and Lemma 11 we have $R(2n)+4=C_{sk-s}(n)+6 \leqq H_{sk-s}(4n)$ $(\leqq H(4n))$. Hence $H_{sk-s}(4n)=H(4n)$. If m is odd and >1, then, by Lemma 12, $H_{sk-s}(m)=4=H(m)$, and, by Lemma 13, we have $5=H_{sk-s}(m)+1 \leqq H_{sk-s}(2m) \leqq H(2m)=5$. Hence $H_{sk-s}(n)=H(n)$ $(=R(8n)-4)$ for all $n>2$. It is obvious that $H_{sk-s}(1)=0$ and $H_{sk-s}(2)=4$. By Theorem 1 we have $R_{sk-s}(n)=R_{sk-h}(n)=R(n)-1$. Thus Theorem 3 is completely proved.

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