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ABELIAN F.P.F. OPERATOR GROUPS OF TYPE (p, p)

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ABSTRACT. A group A of automorphisms on a group G is said to be fixed-point-free, written f.p.f., if $C_G(A)=\{g\in G \mid g^\alpha=g$ for all $\alpha\in A\}=I$. It has been shown by E. Shult that if A is an abelian f.p.f coprime group of automorphisms of order $n=p_1^{a_1}\cdots p_k^{a_k}$ on a solvable group G, then the nilpotent length of G is bounded above by $\psi(n)=\sum_{i=1}^{i=k} a_i$ unless $|G|$ is divisible by primes q such that $q^s+1=d$ where d divides the exponent e of A. F. Gross has removed the exceptional condition on the prime divisors of $|G|$ when A is cyclic of order p^a, p an odd prime. In the case where A is noncyclic of order p^2, the author has also removed the exceptional condition on the prime divisors of $|G|$.

E. Shult [6] has established the following result.

THEOREM. *Let A be an abelian f.p.f. group of automorphisms of order $n=p_1^{a_1}\cdots p_k^{a_k}$ on a solvable group G where $|G|$ is coprime to $|A|$. If $|G|$ is not divisible by primes q such that $q^s+1=d$ where d divides the exponent e of A, then the nilpotent length of G is bounded above by $\psi(n)=\sum_{i=1}^{k} a_i$.*

F. Gross [3] has removed the condition on the prime divisors of $|G|$ when A is cyclic of order p^a, p an odd prime. In this paper, we do the same when A is noncyclic of order p^2, p an arbitrary prime. This result has already been indicated—without proof—by Kurzweil [5]. It should be pointed out that the author has found a counterexample, in the case where A is noncyclic of order p^2, to Theorem 4.1 of Shult [6] when the exceptional condition on the prime divisors of $|G|$ is removed.

1. Preliminaries. All groups under consideration are presumed to be finite.

A group H is said to be an operator group on a group G if there is a homomorphism $\phi: H\to A(G)$ where $A(G)$ is the automorphism group of G. In short, we say that H is an operator group of G. Let $h\phi$ be the image of h under ϕ. We denote the image of g under $h\phi$

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by $g^{h\phi}$, or simply by g^h when we suppress the operator notation. We denote the kernel of the representation as $C_H(G) = \{h \in H \mid g^h = g$ for all $g \in G\}$. Also, we set $C_G(H) = \{g \in G \mid g^h = g$ for all $h \in H\}$. H is said to be fixed-point-free on G, written f.p.f., if $C_G(H) = I$.

We use the following results.

LEMMA 1.1. *Let A be a coprime operator group on a solvable group G. Then, there is an A-invariant Carter subgroup of G.*

LEMMA 1.2. *Let A be an abelian regular group of automorphisms on a group G. Then, A is cyclic.*

LEMMA 1.3. *Let A be a coprime operator group on a group G. If U is a normal A-invariant subgroup of G, then $C_{G/U}(A) = C_G(A)U/U$.*

LEMMA 1.4. *Let A be a coprime operator group on a nilpotent group G. If $\Phi(G)$ is the Frattini subgroup of G and if $\Phi^A(G)$ is the intersection of all the maximal A-invariant subgroups of G, then $\Phi(G) = \Phi^A(G)$.*

LEMMA 1.5. *Let G be the semidirect product of H over K and let ψ be a representation of G on a vector space V/F. If $h \in H$ is contained in the kernel of the representation, then h acts trivially on K/K_1 where K_1 is the kernel of the representation which V affords of K.*

All of the above results are well known except possibly Lemmas 1.1 and 1.5. An adaptation of the argument used to prove Theorem 2.2(i), Chapter 6 in Gorenstein [2], yields Lemma 1.1 and the proof of Lemma 1.5 is trivial.

2. **Main results.** Hereafter, we shall always denote a noncyclic group of order p^2 as a group of type (p, p). Recall that if G is a group, then $O_q(G)$ denotes the largest normal subgroup of G which is a q-group, q a prime.

LEMMA 2.1. *Let $G = \langle \alpha \rangle M$ be a Frobenius group where the Frobenius kernel M is abelian and $|\alpha| = p$. If G has a representation on a vector space V/F where* $\mathrm{char}(F) \nmid |G|$ *and $C_V(\alpha) = (0)$, then $C_V(M) \neq (0)$.*

PROOF. We may assume that F is a splitting field for all subgroups of G.

Let W be an irreducible G-submodule of V. Since $C_W(M) \leqq C_V(M)$, it suffices to consider the case where $W = V$. Now, it follows that if U is an irreducible M-submodule of V, then $U = Fu$ since F is a splitting field for all subgroups of G and M is abelian. Now, if $U < V$, then we have by Satz 17.10, Chapter V in Huppert [4], that $V = \bigoplus \sum_{i=0}^{i=p-1} Fu\alpha^i$. But, this means that $x = \sum_{i=0}^{p-1} u\alpha^i \neq 0$ is fixed by

α, a contradiction to $C_V(\alpha)=(0)$. Thus, it follows that $U=V$. It is now immediate from the linearity of the representation that M must act trivially on $U=V$ since α is regular on M. Therefore, the proof is complete.

LEMMA 2.2. *Let A be a complement of type (p, p) to a normal Hall abelian subgroup H of G. Assume that A contains an element α which is regular on H. If ψ is an absolutely irreducible faithful representation of G on a vector space V/F where* $\text{char}(F)\nmid |G|$, *then* $\deg(\psi)=p$ *if* $C_V(A)=(0)$.

PROOF. Let W be an irreducible H-submodule of V. We have that $W=Fw$ since V is an absolutely irreducible G-module and H is abelian. Also, if H_1 is the kernel of the representation which W affords for H, then $H_1<H$ since G is faithfully and irreducibly represented on V. Now, $W\alpha\neq W$ since otherwise we would have from the linearity of the representation that $[\alpha, H]\leqq H_1<H$, a contradiction to α being regular on H. Consequently, it follows from Satz 17.10, Chapter V in Huppert [**4**], that $U=\bigoplus\sum_{i=0}^{p-1} Fw\alpha^i$ where U is the irreducible $\langle\alpha\rangle H$-submodule of V generated by W. In particular, we have that $\dim_F U=p$. Thus, the proof will be completed when we show that $U=V$. Assume that $U<V$. Then, $\dim_F V>p$. However, this implies by Satz 17.10, cited above, that $\dim_F V=p^2$. This means that $V=\bigoplus\sum_{i=0}^{p-1}\sum_{j=0}^{p-1} Fw\alpha^i\beta^j$ where $A=\langle\alpha, \beta\rangle$. Hence, it follows that $x=\sum_{i=0}^{p-1}\sum_{j=0}^{p-1} w\alpha^i\beta^j\neq 0$ is fixed by all elements of A, a contradiction to the assumption that $C_V(A)=(0)$. Therefore, we are forced to conclude that indeed $\deg(\psi)=p$ and the proof is complete.

THEOREM 2.3. *Let A be a self-normalizing complement of type (p, p) to a normal Hall subgroup H of a solvable group G. Assume that G has a faithful representation on a vector space V/F where $O_q(H)=I$ if* $\text{char}(F)=q\neq 0$. *If A contains an element α such that $C_V(\alpha)=(0)$, then $\alpha\in C_A(H)$. In particular, H is nilpotent.*

PROOF. We argue by induction on $|H|$. The ground case occurs when H is abelian and $H=\times_{i=0}^{k} U^{\beta^i}$, where $A=\langle\alpha, \beta\rangle$ and U is a minimal α-invariant subgroup of H. In particular, U is an elementary abelian r-group where r $(\neq q)$ is a prime. Also, we have by Theorem 2.3, Chapter 6 in Gorenstein [2], that $p\neq q$. It then follows by Theorem 4.4, Chapter 3 in Gorenstein [2], that α centralizes each U^β and consequently H.

Let us assume that H is not nilpotent. Then, the Fitting subgroup $F=F(H)$ of H is properly contained in H. It is clear that the hypotheses of the theorem are satisfied for AF. Thus, we have by induction that $\alpha \in C_A(F)$. We now claim that α acts trivially on $H/C_H(F)$. Let $x \in F$. Then $x^\alpha = x$. Since F is normal in H, it follows that $x^{\alpha g} = x^g \in F$. So, we have $x^g = x^{g\alpha^{-1}} = x^{\alpha g \alpha^{-1}}$. This clearly implies that $[\alpha^{-1}, g^{-1}] \in C_H(F)$. Thus, α acts trivially on $H/C_H(F)$. Now, it is well known that $C_H(F) \leqq F$ since H is solvable. Hence, α acts trivially on H/F. It now follows from the fact that $|\alpha| = p$ is coprime to $|H|$ that α centralizes H. Hence, we may assume that H is nilpotent.

Assume now that H contains two distinct maximal normal A-invariant subgroups, say M_1 and M_2. Then, α centralizes M_i, $i = 1, 2$, by induction and consequently α centralizes H since $H = M_1 M_2$. Thus, we may assume that H contains a unique maximal normal A-invariant subgroup, say M. This says by Lemma 1.4 that $M = \Phi(H)$. Also, we have that H is necessarily an r-group, r a prime, since that Frattini subgroup contains no Hall subgroups of H. Now A cannot be a regular group of automorphisms on H/M by Lemma 1.2. However, A acts irreducibly on H/M; thus, there is a $\gamma \in A^\#$ which centralizes H/M. This implies by Burnside's result—cf., Theorem 1.4, Chapter 5 in Gorenstein [2]—that γ centralizes H since $|\gamma| = p$ is coprime to $|H|$. Also, α centralizes M by induction. Hence, if $\alpha \neq \gamma^i$, then A centralizes M since $\alpha \neq \gamma^i$ implies that $A = \langle \alpha, \gamma \rangle$. But, this contradicts the assumption that A is self-normalizing in G. Therefore, $\gamma_i = \alpha$ and the proof is complete for $\alpha \in C_A(H)$. Now, if $A = \langle \alpha, \beta \rangle$, then β is regular on H since α centralizes H and A is f.p.f. on H. Therefore, H is nilpotent by Thompson's result—cf., Theorem 2.1, Chapter 10 in Gorenstein [2]. This completes the proof of the theorem.

Let A be a group of type (p, p) and let ψ be a representation of A on a vector space V/F such that $C_V(A) = (0)$. If $\text{char}(F) = 0$, then A is completely reducible as an A-module by Maschke's theorem. This means that if V/W is any factor module of V, then $C_{V/W}(A) = (0)$. If $\text{char}(F) = q \neq 0$, then we have by Theorem 2.3, Chapter 6 in Gorenstein [2], that $q \neq p$. Thus, V is completely reducible as an A-module and we have as above that $C_{V/W}(A) = (0)$ for any factor module of V. In any event, $C_{V/W}(A) = (0)$.

THEOREM 2.4. *Let A be a self-normalizing complement of type (p, p) to a normal Hall subgroup H of a solvable group G. Assume that G has a faithful representation on a vector space V/F where $O_q(H) = I$ if* $\text{char}(F) = q \neq 0$. *Then, H is nilpotent if $C_V(A) = (0)$.*

PROOF. We argue by induction on $|G| + \dim_F V$. The ground case is clear.

Let F_1 be a splitting field for all subgroups of G. Set $V_1 = F_1 \otimes_F V$ and consider the representation which V_1 affords for G in the obvious manner. Now, it is easy to see that the hypotheses of the theorem are satisfied for the induced representation. Hence, we may assume that $F_1 = F$.

We first consider the case where G does not act irreducibly on V. Let $(0) < V_1 < V$ be a G-submodule of V. Let H_1 be the kernel of the representation which V_1 affords for H. Now, if $A^\#$ contains an element, say τ, which acts trivially on V_1, then τ acts trivially on H/H_1 by Lemma 1.5. Thus, if $A = \langle \tau, \beta \rangle$, then β is regular on H/H_1 since A is f.p.f. on H/H_1 by Lemma 1.3 and τ acts trivially on H/H_1. Hence, it follows by Thompson's result that H/H_1 is nilpotent. If A is faithful on V_1, then the kernel of the representation which V_1 affords for G is H_1 since $|A|$ is coprime to $|H|$. Thus, we can apply our induction hypothesis to $[G:H_1] + \dim_F V_1$ to obtain that H/H_1 is nilpotent. In either case, we have that H/H_1 is nilpotent. Now, let H_2 be the kernel of the representation which V/V_1 affords for H. By repeating the above arguments, it follows that H/H_2 is nilpotent. This means that $H/H_1 \cap H_2$ is nilpotent. Now, if $H_1 \cap H_2$ contains an element $x \neq 1$ the order of which is not divisible by char(F), then clearly x acts trivially on V since it acts trivially on V_1 and V/V_1. Consequently, it follows that $H_1 \cap H_2 = I$ unless char$(F) = q \neq 0$, since G is faithful of V, in which case $H_1 \cap H_2$ is a q-group. But, $H_1 \cap H_2$ is normal in H and $O_q(H) = I$. Therefore, $H_1 \cap H_2 = I$ in either case which yields the nilpotency of H. We thereupon may assume that G acts irreducibly on V.

Now, if $A^\#$ contains an element α such that $C_V(\alpha) = (0)$, then we have by Theorem 2.3 that H is nilpotent. So, assume that $C_V(\alpha) \neq (0)$ for all $\alpha \in A^\#$.

We now claim that the hypercenter Z_∞ of H is trivial. This is in fact equivalent to the center Z of H being nontrivial. Assume that $Z \neq I$. Since Z is normal in H and since $O_q(H) = I$, it follows that $O_q(Z) = I$. Now, we have by Lemma 1.2 that there is a $\gamma \in A^\#$ such that $C_Z(\gamma) \neq I$. Set $U = C_Z(\gamma)$. Clearly, U is a normal A-invariant subgroup of H. Consider $W = C_V(\gamma)$. We have by the above paragraph that $W \neq (0)$. It is easy to see that W is an AU-submodule of V. In particular, W is a $\langle \sigma \rangle U$-submodule of V where $A = \langle \gamma, \sigma \rangle$. Now, σ is regular on U since γ centralizes U and A acts in a f.p.f. manner on H and hence on U. Also, σ is regular on W for the same reason. It then follows by Lemma 2.1 since q does not divide $|\langle \sigma \rangle U|$ that $C_W(U) \neq (0)$ and hence, $Y = C_V(U) \neq (0)$. But, Y is a G-submodule of V since U is a normal A-invariant subgroup of H. Consequently, it follows from the irreducibility of V that $Y=V$ and we have a contradiction to the assumption that G is faithful on V. Therefore, $Z_\infty=I$ as claimed. This means, of course, that H is not nilpotent.

Now, let L be any proper normal A-invariant subgroup of H. Since $O_q(L)$ is characteristic in L, it is normal in H and so, $O_q(L)=I$. Also, AL is faithful on V. Hence, we can apply our induction hypothesis to $|AL|+\dim_F V$ to obtain that L is nilpotent. Therefore, all proper normal A-invariant subgroups of H are nilpotent.

According to Lemma 1.1 H contains an A-invariant Carter subgroup, say C. Let $K=\Gamma_\infty(H)$ denote the hypercommutator subgroup of H, the smallest normal subgroup of H with nilpotent factor group. We assert that K is a minimal normal A-invariant subgroup of H. Assume that there is a nontrivial normal A-invariant subgroup, say M, which is properly contained in K. If $H=CM$, then it follows that $H/M=CM/M\cong C/C\cap M$ which is nilpotent since all Carter subgroups are nilpotent. But, this contradicts the fact that K is the smallest normal subgroup of H such that H/K is nilpotent. Hence, $CM<H$. Now, we have that $C_H(M)<H$ since $Z_\infty=I$. Since $C_H(M)$ is clearly a normal A-invariant subgroup of H, it follows from the above paragraph that $C_H(M)$ is nilpotent. Hence, q does not divide $|C_H(M)|$ since it is a nilpotent normal subgroup and $O_q(H)=I$. We now consider $O_q(CM)$. Since it is a normal q-subgroup of CM, it follows that $O_q(CM)\leqq F(CM)$, the Fitting subgroup of CM. Let $CM=S_0>S_1>\cdots>S_i=M>\cdots>S_t=I$ be any chief series of CM which contains M as a member. We have by Satz 4.3, Chapter III in Huppert [4], that $F(CM)=\bigcap_{i=0}^{t-1} C_{CM}(S_i/S_{i+1})$. Then, $O_q(CM)\leqq F(CM)$ and q not dividing $|M|$—since M is a nilpotent normal subgroup of H and $O_q(H)=I$—yields by Theorem 3.2, Chapter 5 in Gorenstein, that $O_q(CM)\leqq C_H(M)$. Thus, $O_q(CM)=I$. We then apply our induction hypothesis to $|CM|+\dim_F V$ to obtain that CM is nilpotent. However, this implies that $C<N_{CM}(C)$, a contradiction to the fact that Carter subgroups are self-normalizing subgroups. Therefore, we have established that K is a minimal normal A-invariant subgroup of H.

Now, it follows that $C\cap K=I$ since $H=CK$, H is not nilpotent, and K is a minimal normal A-invariant subgroup of H. We claim that C contains no proper normal A-invariant subgroups. Assume to the contrary and let C_0 be such a subgroup. Now, C_0K is clearly a normal A-invariant subgroup of H and hence, C_0K is nilpotent by previous arguments. This means that $C_K(C_0)\neq I$. But, K is abelian, since it is a minimal normal A-invariant subgroup of H, and C_0 is a normal A-invariant subgroup of C. Thus, it follows from the fact that $H=CK$ that $C_K(C_0)\neq I$ is a normal A-invariant subgroup of H. Thus, $C_K(C_0)=K$ from the minimality of K. This means that C_0 is a normal A-invariant subgroup of H. In fact, this implies by Theorem 11.11, Chapter 6 in Huppert [**4**], that $C_0\leqq Z_\infty$ since C is a system normalizer of H by Theorem 13.4, Chapter 6 in Huppert [**4**]. This obviously contradicts the assumption that $Z_\infty=I$. Therefore, C contains no proper normal A-invariant subgroups as claimed. In particular, we have that C is abelian.

Now, since A acts irreducibly on C and since A cannot be a regular group of automorphisms on C by Lemma 1.2, it follows that $A^\#$ contains an element, say α_0, which centralizes C. Thus, $C\leqq C_H(\alpha_0)$. Now, if $A=\langle\alpha_0, \beta\rangle$, then β is regular on $C_H(\alpha_0)$ since A is f.p.f. on H and hence on $C_H(\alpha_0)$. Hence, it follows by Thompson's result that $C_H(\alpha_0)$ is nilpotent. We then have from the fact that C is self-normalizing in H that $C=C_H(\alpha_0)$. In particular, it follows from this that α_0 is regular on K.

Now, we let U be any irreducible $\langle\alpha_0\rangle K$-submodule of V. Since G acts faithfully and irreducibly on V, it follows that the kernel K_1 of the representation which U affords for K is properly contained in K. Also, α_0 acts nontrivially on U since otherwise we would have by Lemma 1.5 that $[\alpha_0, K]\leqq K_1<K$, a contradiction to α_0 being regular on K. Furthermore, we have by this same line of reasoning that $\dim_F U>1$. It then follows by Satz 17.10, Chapter V in Huppert, that $\dim_F U=p$ since F is a splitting field for all subgroups of G. In addition to this, U is also an AK-submodule by Lemma 2.2. We claim that $U=V$. In this respect, it is necessary and sufficient to show that $Uc=U$ for all $c\in C$ since V is an irreducible G-module. Let $\beta\in A$ where $A=\langle\alpha_0, \beta\rangle$. Since α_0 centralizes C, it is clear that Uc is an $\langle\alpha_0\rangle K$-submodule of V and hence, is an AK-submodule of V by Lemma 2.2 since $\dim_F U=p$. So, we have for each $c\in C$ that $Uc=Uc\beta=U\beta^{-1}c\beta$. This means that $U=U[\beta, c^{-1}]$ for all $c\in C$. Now, $\alpha_0\in C_G(C)$, and so, β is regular on C since A is f.p.f. on C. It is well known that the regularity of β on C guarantees that each element x of C has a unique representation of the form $x=[\beta, c^{-1}]$. Thus, $Uc=U$ which establishes that $U=V$. In particular, we have that $\dim_F V=p$.

Now, let $\{\gamma_1, \cdots, \gamma_{p+1}\}$ be a complete set of generators for the $p+1$ cyclic subgroups of A. Since, $p\neq q$ and $C_V(A)=(0)$, it follows that $V=\bigoplus\sum_{i=1}^{p+1} C_V(\gamma_i)$. Consequently, it follows from $\dim_F V=p$ that there is at least one γ_i such that $C_V(\gamma_i)=(0)$. But, this implies by Theorem 2.3 that H is nilpotent, a contradiction to the assumption that $Z_\infty=I$. Whence, the proof is complete.

We denote the nilpotent length of a solvable group G by $n(G)$. Let $K_i = K_i(G)$ denote the ith member of the descending nilpotent series, that is, K_i is the smallest normal subgroup of G such that $n(G/K_i) \leqq i$. We now prove the main result of this paper.

THEOREM 2.5. *Let A be a group of type (p, p) which is a f.p.f. coprime operator group on a solvable group G. Then, $n(G) \leqq 2$.*

PROOF. Assume that the theorem is false and let G be a counterexample of minimal order.

Consider K_2 and let $L \neq I$ be any normal A-invariant subgroup of G such that $L \leqq K_2$. Now, we have by Lemma 1.3 that A acts in a f.p.f. manner on G/L and so, $n(G/L) \leqq 2$ by the minimality of G. Since K_2 is the smallest normal subgroup of G such that $n(G/K_2) \leqq 2$, it follows that $K_2 \leqq L$. Thus, K_2 contains no nontrivial normal A-invariant subgroups of G properly. In particular, K_2 is an elementary abelian q-subgroup of G, q a prime.

Let K_1 be the hypercommutator subgroup of G and let C be an A-invariant Carter subgroup of K_1. Now, $C < K_1$ since K_1 is not nilpotent. Set $N = N_G(C)$ and let $g \in G$. Then, $C^g < K_1$ since K_1 is normal in G. But, all Carter subgroups of K_1 are conjugate in K_1. So, there is an $x \in K_1$ such that $C^g = C^x$. In fact, we can choose x so that $x \in K_2$ since $K_1 = CK_2$. Thus, it follows that $gx^{-1} \in N$. This means that $G = NK_2$. Now, it is clear that N is an A-invariant subgroup of G since C is an A-invariant subgroup. Thus, $N \cap K_2 = I$ since K_2 is a minimal normal A-invariant subgroup of G and $G = NK_2$.

We now claim that K_2 is self-centralizing in G. This is equivalent to $T = C_N(K_2) = I$. Assume that $T \neq I$. Since T is a normal A-invariant subgroup of N and T centralizes K_2, it follows that T is a normal A-invariant subgroup of G. We then have by the minimality of G that $n(G/T) \leqq 2$. This means that $K_2 \leqq T$ since K_2 is the smallest normal subgroup of G such that $n(G/K_2) \leqq 2$. This clearly contradicts $N \cap K_2 = I$. Thus, $T = I$ as claimed.

Now, set $U = O_q(N)$ and assume that $U \neq I$. Since UK_2 is a q-group, it follows that $C_{K_2}(U) \neq I$. However, U is a normal A-invariant subgroup of N; consequently, $C_{K_2}(U)$ is a normal A-invariant subgroup of G since K_2 is abelian. This means that $K_2 = C_{K_2}(U)$ because of the minimality of K_2. However, this is a contradiction to K_2 being self-centralizing in G. Therefore, $U = O_q(N) = I$.

We now view K_2 as a vector space over Z_q and consider the representation which it affords for the semidirect product AN of A over N. Since N is not nilpotent and since A is f.p.f. on N it follows by Lemma 1.5 that A is faithful on K_2. Hence, it follows that the kernel of the representation is contained in N since $|A|$ is coprime to $|N|$. Thus, the representation is faithful since K_2 is self-centralizing in $G=NK_2$. Finally, A is clearly self-normalizing in AN since A acts in a f.p.f. manner on N. It thereupon follows by Theorem 2.4 that N is nilpotent which contradicts our assumption that the theorem is false, and the proof is complete.

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