REAL-ANALYTIC SUBMANIFOLDS OF COMPLEX MANIFOLDS

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ABSTRACT. This paper examines the extendibility of holomorphic functions on a real manifold which is embedded in a complex manifold. The principal result is that all real k -dimensional, real-analytic, compact manifolds embedded in an n -dimensional complex Stein manifold, where $k > n$, are extendible over a manifold of one higher real dimension. A discussion is also given of the local equations of a manifold which is C-R in a neighborhood of some point.

1. Introduction. Wells [3] proves that there is an open dense set of embeddings of a real k -dimensional $\mathcal{C}^\infty$ compact manifold into an n -dimensional complex manifold X , $k > n$, with the property that each submanifold of X embedded by an element in this dense set is extendible over a real $(k+1)$ -dimensional $\mathcal{C}^m$ manifold, $1 \leq m < \infty$. It is the purpose of this paper to show that all real k -dimensional, real-analytic, compact manifolds embedded in an n -dimensional complex Stein manifold X , $k > n$, are extendible over a manifold of one higher real dimension.

In §2 of this paper we give definitions of exceptional points, extendibility, and the Levi form. §3 contains a discussion of the local equations of a manifold which is C-R in a neighborhood of some point. A short discussion of peak points is given in §4, and we prove our result for real-analytic submanifolds of complex manifolds in §5.

2. Definitions. Let M^k be a real k -dimensional $\mathcal{C}^\infty$ manifold embedded in an n -dimensional complex manifold X , where $k, n \geq 2$. If f is the embedding, then the real Jacobian of f has maximal rank. Let us denote the complex Jacobian of f by $J(f)$ and set $q = \min(n, k)$. A point p in M^k is said to be an *exceptional point of order l* , $0 \leq l \leq [k/2] - \max(k-n, 0)$, if the complex rank of $J(f)|_p$ is equal to $q-l$.

A point p in M^k is *generic* if p is an exceptional point of order 0. The manifold M^k is *locally generic* at p in M^k if every point in some open neighborhood of p is generic, and is *locally C-R* at p if every point in some open neighborhood of p is an exceptional point of the

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same order. If $k \leq n$, an exceptional point of order 0 is a *totally real* point.

Suppose M^k is locally C-R at p , and p is not a totally real point for M^k . Let $T_x(M^k)$ be the real tangent space to M^k at x , and $H_x(M^k)$ be the space of holomorphic tangent vectors to M^k at x . Then we define the Levi form, at any x near p ,

$$L_x(M^k): H_x(M^k) \rightarrow (T_x(M^k) \otimes C)/(H_x(M^k) \otimes C)$$

by $L_x(M^k)(t) = \pi_x \{ [Y, \bar{Y}]_x \}$, where Y is a local section of $H(M^k)$ such that $Y_x = t$, $[Y, \bar{Y}]_x$ is the Lie bracket evaluated at x , and

$$\pi_x: T_x(M^k) \otimes C \rightarrow (T_x(M^k) \otimes C)/(H_x(M^k) \otimes C)$$

is the projection.

Denote by $\mathcal{O}_x = \mathcal{O}$ the sheaf of germs of holomorphic functions on the second countable complex manifold X . Let K be a compact subset of X and U an open subset of X containing K . We set $\mathcal{O}(K) = \text{ind} \lim_{U \supset K} \mathcal{O}(U)$ where $\mathcal{O}(U)$ is the Fréchet algebra of holomorphic functions on U . We say that K is extendible to a connected set $K' \supsetneq K$ if the map $r: \mathcal{O}(K') \rightarrow \mathcal{O}(K)$ is onto.

3. Local equations. Let $J^r(M^k, X)$ be the r -jet bundle with fiber $J^r(k, n)$ (e.g. see [2]). We can identify $J^1(k, n)$ with the set of all complex $(n \times k)$ matrices. Denote by $S_l(k, n) = S_l$ the subset of $J^1(k, n)$ with rank equal to $(q-l)$, $0 \leq l \leq [k/2] - \max(k-n, 0)$. Then if M^k is embedded in X by the map f , a point p in M^k is an exceptional point of order l if and only if $J(f)|_p$ is in S_l .

Suppose the point p in M^k is an exceptional point of order l . If $k \geq n$, then the local equations of M^k in a neighborhood of p are given (after a proper coordinate change) by

$$\begin{aligned}
 (1) \quad & z_1 = x_1 + ih_1(x_1, \dots, x_{2(n-l)-k}, w_1, \dots, w_{k-n+l}), \\
 & \vdots \\
 & z_{2(n-l)-k} = x_{2(n-l)-k} + ih_{2(n-l)-k}(x_1, \dots, x_{2(n-l)-k}, w_1, \dots, w_{k-n+l}), \\
 & z_{2(n-l)-k+1} = u_1 + iv_1 = w_1, \\
 & \vdots \\
 & z_{n-l} = u_{k-n+l} + iv_{k-n+l} = w_{k-n+l}, \\
 & z_{n-l+1} = g_1(x_1, \dots, x_{2(n-l)-k}, w_1, \dots, w_{k-n+l}), \\
 & \vdots \\
 & z_n = g_l(x_1, \dots, x_{2(n-l)-k}, w_1, \dots, w_{k-n+l}),
 \end{aligned}$$

where $x_1, \dots, x_{2(n-l)-k}, u_1, v_1, \dots, u_{k-n+l}, v_{k-n+l}$ are local coordinates for M^k in a neighborhood of p vanishing at p , and $z_1, \dots, z_n$ are local coordinates for X in a neighborhood of p vanishing at p . The real-valued functions $h_1, \dots, h_{2(n-l)-k}$ as well as the complex-valued functions $g_1, \dots, g_l$ vanish to order 2 at p .

In [2] we state that S_l is a regular submanifold of $J^1(k, n)$ of complex codimension $(k-n+l)l$. However, we notice that $(k-n+l)l$ is just the number of complex lines in the local equations times the number of functions g_i in the local equations.

The question we are interested in is the following. What form must the equations take in order for the embedded manifold to be locally C-R at the exceptional point p of order l ? Since every point p' in some open neighborhood of p would have the property that $J(f)|_{p'}$ must satisfy $(k-n+l)l$ complex conditions, we find it is difficult (in the sense of transversality theory) for a manifold to be locally C-R except when $l=0$. Using equations (1) we find that our complex Jacobian at a point p' can be put in the form

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix},$$

where A is the $(2(n-l)-k) \times (2(n-l)-k)$ identity matrix, B and C are both zero matrices, and D is the $(2l+k-n) \times (2(k-n+l))$ matrix

$$\begin{bmatrix} 1 & i & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & i & \dots & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 1 & i \\ \frac{\partial g_1}{\partial u_1} & \frac{\partial g_1}{\partial v_1} & \frac{\partial g_1}{\partial u_2} & \frac{\partial g_1}{\partial v_2} & \dots & \frac{\partial g_1}{\partial u_{k-n+l}} & \frac{\partial g_1}{\partial v_{k-n+l}} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{\partial g_l}{\partial u_1} & \frac{\partial g_l}{\partial v_1} & \frac{\partial g_l}{\partial u_2} & \frac{\partial g_l}{\partial v_2} & \dots & \frac{\partial g_l}{\partial u_{k-n+l}} & \frac{\partial g_l}{\partial v_{k-n+l}} \end{bmatrix}.$$

Hence M^k is locally C-R at the exceptional point p of order l if and only if $(\partial g_i / \partial u_j) = -i(\partial g_i / \partial v_j)$ holds in some neighborhood of p for $j=1, \dots, k-n+l$ and $i=1, \dots, l$. Thus for the manifold M^k to be locally C-R the functions $g_1, \dots, g_l$ in our local equations (1) must be complex-analytic functions of $w_1, \dots, w_{k-n+l}$.

Again let p be an exceptional point of order l in M^k , but assume $k < n$. The local equations of M^k in a neighborhood of p are given (after a proper coordinate change) by

$$\begin{aligned}
 (2) \quad & \begin{aligned} & z_1 = x_1 + ih_1(x_1, \dots, x_{k-2l}, w_1, \dots, w_l), \\ & \vdots \\ & z_{k-2l} = x_{k-2l} + ih_{k-2l}(x_1, \dots, x_{k-2l}, w_1, \dots, w_l), \\ & z_{k-2l+1} = u_1 + iv_1 = w_1, \\ & \vdots \\ & z_{k-l} = u_l + iv_l = w_l, \\ & z_{k-l+1} = g_1(x_1, \dots, x_{k-2l}, w_1, \dots, w_l), \\ & \vdots \\ & z_n = g_{n-k+l}(x_1, \dots, x_{k-2l}, w_1, \dots, w_l), \end{aligned}
 \end{aligned}$$

where $x_1, \dots, x_{k-2l}, u_1, v_1, \dots, u_l, v_l$ are the local coordinates for M^k in a neighborhood of p vanishing at p , and $z_1, \dots, z_n$ are the local coordinates for X in a neighborhood of p vanishing at p .

We have that S_l is a regular submanifold of $J^1(k, n)$ of complex codimension $(n-k+l)l$. In order for the manifold to be locally C-R at p , a discussion similar to the previous one tells us that the functions $g_1, \dots, g_{n-k+l}$ in our local equations (2) must be complex-analytic functions of $w_1, \dots, w_l$.

4. Peak points and the Levi form. Let K be a compact subset of a complex manifold X . We call a point $x \in K$ a *holomorphic peak point* if there exists a function $f \in \mathcal{O}(K)$ such that, for any $y \in K - \{x\}$, we have $|f(y)| < |f(x)|$.

In [3] it is shown that a compact $\mathbb{C}^2$ submanifold of a Stein manifold X contains a nonempty open set of peak points. Also the Levi form cannot vanish at a peak point without contradicting the maximum principle.

5. Real-analytic manifolds. In this section we shall prove the asserted result for real-analytic manifolds.

THEOREM 5.1. *Let M^k be a real k -dimensional, real-analytic, compact manifold embedded in an n -dimensional Stein manifold X , $k > n$. Then M^k is extendible to a $\mathbb{C}^m$ submanifold $\tilde{M}$ of X with real dimension $\tilde{M} = k+1$ and $1 \leq m < \infty$.*

PROOF. Given any point p in M^k , arbitrarily close to p we can find an open set of points at which M^k is locally C-R. This follows from

the fact that if M^k is not locally C-R at p , then arbitrarily close to p we can find a point which is an exceptional point of order less than the order of p . We repeat this process until we find a point at which M^k is locally C-R (for a "dense" set of embeddings this will mean locally generic).

Let p be a peak point in the open set mentioned in §4. Then there exists a point p' in M^k and an open neighborhood of this point on which M^k is locally C-R and the Levi form does not vanish. If M^k is locally generic at p' , we have the conditions for local extendibility given in [1] and [4] and the proof is complete. If M^k is locally C-R at p' and not generic there we have the local equations (1) where $l > 0$ and $g_1, \dots, g_l$ are complex-analytic functions of $w_1, \dots, w_{k-n+l}$.

Since our manifold is real-analytic, we expand each function g_i in a power series

$$g_i(x, w) = \sum_{\alpha} a_{i,\alpha}(x, w)^{\alpha},$$

where $x = (x_1, \dots, x_{2(n-l)-k})$, $w = (w_1, \dots, w_{k-n+l})$, and $\alpha = (\alpha_1, \dots, \alpha_{n-l})$. Replacing z_j by $z_j - \sum_{\alpha} a_{i,\alpha} z^{\alpha}$, where $z = (z_1, \dots, z_{n-l})$, $j = n-l+1, \dots, n$, and $j = n-l+i$, we find in our new coordinates that each $a_{i,\alpha} = 0$. Hence our local equations in our new coordinate system are

$$\begin{aligned} (3) \quad & \begin{aligned} & z_1 = x_1 + i h_1(x_1, \dots, x_{2(n-l)-k}, w_1, \dots, w_{k-n+l}), \\ & \vdots \\ & z_{2(n-l)-k} = x_{2(n-l)-k} + i h_{2(n-l)-k}(x_1, \dots, x_{2(n-l)-k}, w_1, \dots, w_{k-n+l}), \\ & z_{2(n-l)-k+1} = u_1 + i v_1 = w_1, \\ & \vdots \\ & z_{n-l} = u_{k-n+l} + i v_{k-n+l} = w_{k-n+l}, \\ & z_{n-l+1} = 0, \\ & \vdots \\ & z_n = 0. \end{aligned} \end{aligned}$$

Thus our point p' appears locally as a generic point of a real k -dimensional submanifold of C^{n-l} . Since the Levi form's vanishing or nonvanishing is independent of coordinate changes, we have a generic point at which the Levi form does not vanish. This establishes the extendibility as given in the statement of the theorem.

REMARK. The theorem is false if $k \leq n$ because of the existence of totally real submanifolds which are always holomorphically convex.

REFERENCES

1. S. J. Greenfield, *Cauchy-Riemann equations in several variables*, Ann. Scuola Norm. Sup. Pisa (3) **22** (1968), 275–314. MR **38** #6097.
2. L. R. Hunt, *Exceptional points of a differentiable submanifold of a complex manifold* (to appear).
3. R. O. Wells, Jr., *Concerning the envelope of holomorphy of a compact differentiable submanifold of a complex manifold*, Ann. Scuola Norm. Sup. Pisa (3) **23** (1969), 347–361. MR **39** #7141.
4. ———, *Holomorphic hulls and holomorphic convexity of differentiable submanifolds*, Trans. Amer. Math. Soc. **132** (1968), 245–262. MR **36** #5392.

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