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A NOTE ON SOBOLEV ALGEBRAS

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ABSTRACT. Sufficient conditions are given for the Sobolev space $L_w^p=\{f\in L^p(E^n):\mathfrak{F}^{-1}(\hat{f}(\xi)w(\xi))\in L^p\}$ to form an algebra under pointwise multiplication, when $1\leqq p\leqq 2$. The conditions are verified for some examples.

In a previous work [6] we showed that the Sobolev space $L_\alpha^p=\{f\in L^p(E^n):\mathfrak{F}^{-1}(\hat{f}(\xi)(1+|\xi|^2)^{\alpha/2})\in L^p\}$ forms an algebra under pointwise multiplication for $1<p<\infty$ provided $\alpha>n/p$. The condition $\alpha>n/p$ is exactly what is required to have the containment $L_\alpha^p\subseteq L^\infty$. Thus it is reasonable to conjecture that for a wide class of weight functions $w(\xi)$ (replacing $(1+|\xi|^2)^{\alpha/2}$), the space $L_w^p=\{f\in L^p(E^n):\mathfrak{F}^{-1}(\hat{f}(\xi)w(\xi))\in L^p\}$ should form an algebra provided $L_w^p\subseteq L^\infty$. This has been established by Bagby [1] for weight functions of the form $w(\xi)=(1+i\xi_1+\sum_{j=2}^n\xi_j^2)^\alpha$ by generalizing our methods in [6]. The intricacy of his proof, however, indicates that other methods should be sought to handle more general weight functions, such as

$$w(\xi)=\left(1+\sum_{j=1}^n\xi_j^{2m_j}\right)^\alpha, \tag{1}$$

$m_1,\cdots,m_n$ positive integers, considered in Sadosky and Cotlar [4].

In this note we give such a method which works in the case $1\leqq p\leqq 2$. Since we require $L_w^p\subseteq A$, the space of Fourier transforms of L^1 functions, there does not seem to be an extension to $p>2$.

Denote by M_p the space of L^p multipliers; $M_p=\{m(\xi)\in L^\infty(E^n):\mathfrak{F}^{-1}(m(\xi)\hat{f}(\xi))$ is a bounded operator on $L^p\}$ equipped with the operator norm.

THEOREM. *Let* $1\leqq p\leqq 2$. *Assume*

(a) $w^{-1}\in M_p\cap L^p$,

(b) $\sup_\eta\|w(\xi)/(w(\xi-\eta)+w(\eta)):M_p\|=M<\infty$.

Then L_w^p *is an algebra under pointwise multiplication.*

PROOF. Let $f,\ g\in L_w^p$. We must show that $\mathfrak{F}^{-1}(\hat{f}*\hat{g}(\xi)w(\xi))\in L^p$. If we write $a(\xi,\eta)=w(\xi)/(w(\xi-\eta)+w(\eta))$ we have

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$$\mathfrak{F}^{-1}(\hat{f} * \hat{g}(\xi)w(\xi)) = \mathfrak{F}^{-1}\left(\int \hat{f}(\xi - \eta)\hat{g}(\eta)a(\xi, \eta)w(\eta)\, d\eta\right) + \mathfrak{F}^{-1}\left(\int \hat{f}(\xi - \eta)\hat{g}(\eta)a(\xi, \eta)w(\xi - \eta)\, d\eta\right). \tag{2}$$

We show that each term separately is in L^p.

Note that $w^{-1} \in L^p$ implies trivially that $L^p_w \subseteq A$, in other words $\hat{f}, \hat{g} \in L^1$. Thus

$$\left\| \mathfrak{F}^{-1}\left(\int \hat{f}(\xi - \eta)\hat{g}(\eta)a(\xi, \eta)w(\xi - \eta)\, d\eta\right)\right\|_p \leqq \int \|\mathfrak{F}^{-1}(\hat{f}(\xi - \eta)w(\xi - \eta)a(\xi, \eta))\|_p \,|\hat{g}(\eta)|\, d\eta \leqq M\|f : L^p_w\| \, \|\hat{g}\|_1$$

by assumption (b). To handle the other term in (2) we make a change of variable $\eta \to \xi - \eta$ and repeat the argument interchanging f and g. Q.E.D.

Let us show that all functions of the form (1) with $\alpha > 0$ satisfy condition (b). Condition (a) is satisfied provided $\alpha > (2p)^{-1} \sum_{j=1}^{n} m_j^{-1}$ (see [4]), when $L^p_w \subseteq L^\infty$.

By the Marcinkiewicz multiplier theorem (see [2], [3], or [5]) (b) will follow if we can show

$$|\xi^{\alpha} D_\xi^{\alpha} a(\xi, \eta)| \leqq M \tag{3}$$

for all multi-indices $\alpha = (\alpha_1, \cdots, \alpha_n)$ such that $\alpha_j = 0$ or 1. This may be established by straightforward but tedious estimates using the elementary inequality

$$\sum_{j=1}^{n} \xi_j^{2m_j} \leqq M \max\left\{ \sum_{j=1}^{n} (\xi_j - \eta_j)^{2m_j}, \sum_{j=1}^{n} \eta_j^{2m_j} \right\} \tag{4}$$

if $M \geqq 2^{2m_j+1}$, $j = 1, \cdots, n$.

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