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REMARKS ON SOME TAUBERIAN THEOREMS OF MEYER-KÖNIG, TIETZ AND STIEGLITZ

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ABSTRACT. A general theorem is proved which deduces from a Tauberian condition T_1 for a summability method, Tauberian condition T_2 for the summability method. Recent results of Meyer-König and Tietz and of Stieglitz are special cases.

1. **Introduction.** Let $\sum_{n=0}^{\infty} a_n$ be an infinite series with $s_n = \sum_{k=0}^{n} a_k$, $n \geqq 0$, its partial sums; and let V be a sequence-to-sequence summability method. A condition on the sequence $\{a_n\}$ $(n \geqq 0)$ is called a Tauberian condition for V if its fulfillment by $\{a_n\}$ together with the existence of lim Vs, where $s = \{s_n\}$, implies $\sum_{n=0}^{\infty} a_n$ exists and $= \lim Vs$.

Recently in a series of papers Meyer-König and Tietz [2], [3], [4] and Stieglitz [5] have shown in different ways, methods by which one Tauberian condition may be deduced from another. They have introduced some new Tauberian conditions as well as obtained some known Tauberian conditions for some of the better known summability methods.

This paper will attempt to bridge the results of Meyer-König and Tietz and those of Stieglitz by showing that all are included in a general theorem. This theorem will also enable us to introduce new Tauberian conditions for well-known methods such as the Borel transform.

Finally we wish to express our thanks to Professor W. Meyer-König and to Drs. H. Tietz and M. Stieglitz for letting us see their not yet published papers. We follow here Steiglitz' notation.

2. **Definitions and main results.** Let S denote the sequence-to-sequence transformation transforming the sequence $a = \{a_n\}$ $(n \geqq 0)$ into the sequence of partial sums of the series $\sum_{n=0}^{\infty} a_n$, i.e. $(Sa)_n = \sum_{k=0}^{n} a_k$; and let c_0, c and m denote the space of sequences converging to 0, converging and bounded, respectively. We shall deal with linear Tauberian conditions; thus they will be given by a linear operator T defined on a subspace of the space of sequences. If α denotes one of the spaces c_0, c and m, let (see [5])

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$$\alpha T = \{a = \{a_n\} : Ta \text{ is defined and } \in \alpha\},$$

and finally set $V^* = c(VS)$.

The following definition of a Tauberian operator was introduced by Stieglitz [5].

DEFINITION. Let α be one of c_0, c and m. An operator T is called a Tauberian operator of type α for the method V, in short T^α, if $V^* \cap \alpha T \subseteq cS$, with $\lim Sa = V\text{-}\lim Sa$, $a \in V^* \cap \alpha T$.

Our first result generalizes [5, Satz 1] and reduces in special cases to results of Meyer-König and Tietz [4].

THEOREM 1. *Let V be a regular additive summability method and let α and β denote any of c_0, c and m. Suppose that $T_1 = T_1^\alpha$ and that T_1 possesses a right inverse T_1^{-1} satisfying*

$$(2.1) \qquad T_1(T_1^{-1}a) = a, \qquad a \in \alpha.$$

Suppose that A and B are two sequence-to-sequence transformations satisfying

$$(2.2) \qquad \beta \subseteq cB \cap \alpha A,$$

and that T_2 satisfies

$$(2.3) \qquad Sa = S(T_1^{-1}[A(T_2a)]) + B(T_2a), \qquad a \in V^* \cap \beta T_2.$$

Then $T_2 = T_2^\beta$.

PROOF. First we prove that $V^* \cap \beta T_2 \subseteq cS$. To this end let $a \in V^* \cap \beta T_2$, then $T_2 a \in \beta$ and it follows by (2.2) that $B(T_2a) \in c$. Hence by the regularity of V, V-lim $B(T_2a)$ exists. Since $a \in V^*$ and V is additive, we thus conclude by (2.3) that V-lim $S(T_1^{-1}[A(T_2a)])$ exists. Put now $b = T^{-1}[A(T_2a)]$, then by (2.1) and (2.2), $T_1 b = A(T_2a) \in \alpha$. Given $T_1 = T_1^\alpha$ it follows that lim Sb exists and that lim $Sb = V$-lim Sb, therefore by (2.3) lim Sa exists. Also

$$\begin{aligned}\lim Sa &= \lim Sb + \lim B(T_2a) = V\text{-}\lim Sb + V\text{-}\lim B(T_2a)\\ &= V\text{-}\lim\,[Sb + B(T_2a)] = V\text{-}\lim Sa.\end{aligned}$$

This completes our proof.

We shall derive now the results of Meyer-König and Tietz [4, Sätze 2.1–2.12] from our general theorem. We shall do it briefly and leave the details for the reader.

Let $\{\lambda_n\}$, $\{p_n\}$ and $\{q_n\}$ be sequences with nonzero entries and denote, for $n \geqq 0$,

$$f_n = \frac{q_{n-1}}{p_n \lambda_n}, \qquad g_n = \lambda_n q_n \left(\frac{1}{p_n \lambda_n} - \frac{1}{p_{n+1}\lambda_{n+1}}\right),$$

$$h_n = \frac{q_n - q_{n-1}}{p_n}, \quad \text{and} \quad R = \sum_{n=0}^{\infty} |f_n - f_{n+1}|$$

(where $q_{-1} = 0$). If

(2.4) $$(T_1 a)_n = \lambda_n a_n$$

and

(2.5) $$(T_2 a)_n = \frac{1}{q_n} \sum_{k=0}^{n} p_k \lambda_k a_k,$$

then it follows by [4, (2.4)] that

(2.6) $$(Sa)_n = \sum_{k=0}^{n} \frac{h_k}{\lambda_k} (T_2 a)_k + \sum_{k=0}^{n-1} (f_k - f_{k+1})(T_2 a)_k + f_n (T_2 a)_n.$$

Comparing (2.6) with (2.3) we see that A is the diagonal matrix diag $\{h_n\}$ while $B = \|b_{nk}\|$ where

$$\begin{aligned} b_{nk} &= f_k - f_{k+1}, & 0 \leqq k < n, \\ &= f_n, & k = n, \\ &= 0, & k > n. \end{aligned}$$

Now, B is conservative, that is, it transforms c into c if and only if $R < \infty$, and it transforms m into c if and only if $R < \infty$ and $f_n = o(1)$. As for A, it transforms c_0 into c_0 if and only if $h_n = O(1)$, it transforms c into c if and only if $\{h_n\}$ converges, it transforms m into m if and only if $h_n = O(1)$, and it transforms m into c_0 if and only if $h_n = o(1)$.

Using [4, (2.5)], it follows that

(2.7) $$(Sa)_n = \sum_{k=0}^{n} \frac{g_k}{\lambda_k} (T_2 a)_k + f_{n+1} (T_2 a)_n.$$

In this case we thus have a matrix A which is the diagonal matrix diag $\{g_n\}$ and B is the diagonal matrix diag $\{f_{n+1}\}$. So B transforms c_0 into c_0 if and only if $f_n = O(1)$, it transforms c into c if and only if $\{f_n\}$ converges, and it transforms m into c_0 if and only if $f_n = o(1)$. As for A the cases are similar to those of the above A and obtained by replacing each h_n by g_n.

REMARK. If, in Theorem 1, V is assumed to be conservative rather than regular, we can still prove that $V^* \cap \beta T_2 \subseteq cS$ but not that lim Sa $= V$-lim Sa. In fact this conclusion holds even if we assume $V^* \cap \alpha T_1 \subseteq cS$ rather than $T_1 = T_1^{\alpha}$.

A similar theorem whose proof is left to the reader generalizes [5, Satz 2].

THEOREM 2. *Let V be an additive summability method that transforms* c_0 *into* c_0, *and let* α *and* β *be any of* c_0, c *and* m. *Suppose that* T_1 *satisfies the requirements of Theorem* 1, *that A and B satisfy*

$$\beta \subseteq c_0 B \cap \alpha A, \tag{2.8}$$

and that T_2 *satisfies* (2.3). *Then* $T_2 = T_2^{\beta}$.

As an application one may take the operators T_1 and T_2 given by (2.4) and (2.5) respectively. Then the following is an immediate consequence of (2.7) and Theorem 2.

COROLLARY 1. *Let V be an additive summability method transforming* c_0 *into* c_0. *Then* (i) *if* $g_n = O(1)$ *and* $f_n = O(1)$, *and if* $T_1 = T_1^{c_0}$, *then* $T_2 = T_2^{c_0}$; (ii) *if* $g_n = o(1)$ *and* $f_n = o(1)$, *and if* $T_1 = T_1^{c_0}$, *then* $T_2 = T_2^{m}$; *and* (iii) *if* $g_n = O(1)$ *and* $f_n = o(1)$, *and if* $T_1 = T_1^{m}$, *then* $T_2 = T_2^{m}$.

REMARK. In Theorem 2 if V is assumed to transform c_0 into c rather than c_0 we can still prove that $V^* \cap \beta T_2 \subseteq cS$ but not that $\lim Sa = V\text{-}\lim Sa$. This conclusion holds even if we assume $V^* \cap \alpha T_1 \subseteq cS$ rather than $T_1 = T_1^{\alpha}$.

3. **Some new Tauberian conditions.** Let $\{\lambda_n\}$, $\{p_n\}$ and $\{q_n\}$ be sequences with nonzero entries and let T_1 be the operator defined by (2.4). Define an operator T_2 by taking the transposed matrix of the matrix in (2.5), namely, define

$$(T_2 a)_n = p_n \sum_{k=n}^{\infty} \frac{\lambda_k a_k}{q_k}, \tag{3.1}$$

provided the sum on the right-hand side exists. Denote, for $n \geqq 0$,

$$f_n' = \frac{q_n}{\lambda_n p_{n+1}}, \qquad g_n' = \frac{\lambda_n}{p_n}\left(\frac{q_n}{\lambda_n} - \frac{q_{n-1}}{\lambda_{n-1}}\right),$$

$$h_n' = q_n\left[\frac{1}{p_n} - \frac{1}{p_{n+1}}\right], \quad \text{and} \quad R' = \sum_{n=0} |f_n' - f_{n-1}'|$$

(where $q_{-1} = 0$). The following are proved exactly as (2.6) and (2.7). If for $\{a_n\}$, $T_2 a$ is defined, then

$$(Sa)_n = \sum_{k=0}^{n} \frac{h_k'}{\lambda_k}(T_2 a)_k + \sum_{k=0}^{n} (f_k' - f_{k-1}')(T_2 a)_k - f_n'(T_2 a)_{n+1} \tag{3.2}$$

and

$$(Sa)_n = \sum_{k=0}^{n} \frac{g_k'}{\lambda_k}(T_2 a)_k - f_n'(T_2 a)_{n+1}. \tag{3.3}$$

Therefore parallel to [4, Sätze 2.1–2.12] we have

THEOREM 3. *Let V be a regular additive summability method. Then in what follows if (*) is satisfied and $T_1 = T_1^\alpha$, then $T_2 = T_2^\beta$.*

		(*)	α	β
(i)	$h_n' = O(1)$,	$R' < \infty$,	c_0	c_0
(ii)	$h_n' = o(1)$,	$R' < \infty$,	c_0	c
(iii)	$\{h_n'\} \in c$,	$R' < \infty$,	c	c
(iv)	$h_n' = O(1)$,	$R' < \infty$,	m	c
(v)	$h_n' = o(1)$,	$R' < \infty$, $f_n' = o(1)$,	c_0	m
(vi)	$h_n' = O(1)$,	$R' < \infty$, $f_n' = o(1)$,	m	m
(vii)	$g_n' = O(1)$,	$f_n' = O(1)$,	c_0	c_0
(viii)	$g_n' = o(1)$,	$\{f_n'\} \in c$,	c_0	c
(ix)	$\{g_n'\} \in c$,	$\{f_n'\} \in c$,	c	c
(x)	$g_n' = O(1)$,	$\{f_n'\} \in c$,	m	c
(xi)	$g_n' = o(1)$,	$f_n' = o(1)$,	c_0	m
(xii)	$g_n' = O(1)$,	$f_n' = o(1)$,	m	m

A result similar to Corollary 1 can also be proved.

As an application take any V such that $n^{1/2} a_n = o(1)$ is a Tauberian condition for V; for instance the Borel transform is such a method. Then

$$e^{n^{1/2}} \sum_{k=n}^{\infty} e^{-k^{1/2}} a_k = o(1) \tag{3.4}$$

is also a Tauberian condition for V. For if $p_n = e^{n^{1/2}}$, $q_n = n^{1/2} e^{n^{1/2}}$ and $\lambda_n = n^{1/2}$, $n \geqq 0$ (the values for q_0 and λ_0 are not important), then $h_n' = O(1)$ and $f_n' \uparrow 1$, as $n \to \infty$, whence $R' < \infty$. Our result follows now by Theorem 3(i).

Take V such that $\lambda_n a_n = o(1)$ is a Tauberian condition for V. Then if $\lambda_n = o(n)$ or $\lambda_n = O(n)$, then $\{n \sum_{m=n}^{\infty} a_m/m\}$ converges or $n \sum_{m=n}^{\infty} a_m/m = o(1)$, respectively is also a Tauberian condition for V. For if $p_n = n$ and $q_n = n\lambda_n$, then $h_n' = o(1)$ or $h_n' = O(1)$ respectively and $f_n' \uparrow 1$, as $n \to \infty$, whence $R' < \infty$. Therefore our results follow by Theorems 3(iii) and 3(i).

As the case $\lambda_n = n^{1/2}$ shows, it is possible to obtain different Tauberian operators T_2 even if we start with the same T_1. We shall give still another Tauberian operator, an operator that was used by us

[1] while dealing with the quasi-Hausdorff methods. It is related also to the operators defined by [4, (2.9)] and [5, (9)].

Let T_1 be the operator defined by (2.4) where $\lambda_n \neq 0, -1$, $n=0, 1, 2, \cdots$. Define the operator T_2 by

$$(T_2 a)_n = \sigma_n \sum_{k=n}^{\infty} \frac{a_k}{\sigma_{k+1}}, \tag{3.5}$$

where

$$\begin{aligned} \sigma_n &= 1, && n = 0, \\ &= \prod_{m=0}^{n-1} \left(1 + \frac{1}{\lambda_m}\right), && n \geqq 1, \end{aligned}$$

whenever the sum on the right-hand side of (3.5) exists. It is readily seen that if $T_2 a$ is defined, then

$$(Sa)_n = \sum_{k=0}^{n} \frac{1}{\lambda_k} (T_2 a)_k + (T_2 a)_0 - (T_2 a)_{n+1}. \tag{3.6}$$

Consequently A is the identity operator while $B = \|b_{nk}\|$ where

$$\begin{aligned} b_{nk} &= 1, && n \geqq 0, \quad k = 0, \\ &= -1, && k = n + 1, \\ &= 0, && \text{elsewhere.} \end{aligned}$$

Hence B transforms c_0 into c and c into c. The following is an immediate consequence of Theorem 1.

THEOREM 4. *Let V be a regular additive summability method and let T_1 and T_2 be defined by* (2.4) *and* (3.5) *respectively. Then if $T_1 = T_1^{c_0}$, then $T_2 = T_2^{c_0}$ and if $T_1 = T_1^{c}$, then $T_2 = T_2^{c}$.*

For some quasi-Hausdorff methods $T_1 = T_1^{c_0}$ and so also $T_2 = T_2^{c_0}$ as can be obtained by [1].

4. **Conclusion.** Let the operator T_1 be a Tauberian operator of type α, for the regular additive method V, which possesses a right inverse T_1^{-1} satisfying (2.1). Let β be one of c_0, c and m. Then we may ask whether it is possible to find T_2 a Tauberian operator of type β for V.

If there exist operators A and B which satisfy (2.2) and such that $(ST_1^{-1}A + B)$ possesses a nontrivial right inverse $(ST_1^{-1}A + B)^{-1}$ with

$$(ST_1^{-1}A + B)[(ST_1^{-1}A + B)^{-1}a] = a, \tag{4.1}$$

for all a such that $(ST_1^{-1}A + B)^{-1}a \in \beta$, then T_2 may be defined by

$$T_2 = (ST_1^{-1}A + B)^{-1}S. \tag{4.2}$$

For then T_2 satisfies (2.3) and our result follows by Theorem 1.
Similar remarks can be made if V transforms c_0 into c_0.

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