ON PARACOMPACTNESS IN FUNCTION SPACES WITH THE COMPACT-OPEN TOPOLOGY

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ABSTRACT. A k -network $\mathcal{O}$ for a space X is a family of subsets of X such that if $C \subset U$, with C compact and U open, then there is a finite union R of members of $\mathcal{O}$ such that $C \subset R \subset U$. An $\mathbf{N}_0$ -space is a T_2 -space having a countable k -network and an $\mathbf{N}$ -space is a T_2 -space having a σ -locally finite k -network. In this paper, it is shown that if X is an $\mathbf{N}_0$ -space and Y is a paracompact $\mathbf{N}$ -space, then $\mathcal{C}(X, Y)$, with the compact-open topology is a paracompact $\mathbf{N}$ -space. The result implies that if X is separable metric and Y is metric, then $\mathcal{C}(X, Y)$ is paracompact.

The purpose of this paper is to show that certain function spaces are paracompact. We shall introduce a class of spaces called $\mathbf{N}$ -spaces which share many of the properties of E. Michael's $\mathbf{N}_0$ -spaces [3], though an $\mathbf{N}$ -space need not be Lindelöf. The $\mathbf{N}_0$ -spaces can be characterized as Lindelöf or hereditarily separable $\mathbf{N}$ -spaces [7]. Throughout, $\mathcal{C}(X, Y)$ will denote the space of continuous functions from X to Y equipped with the compact-open topology. Michael has shown that if X and Y are $\mathbf{N}_0$ -spaces, then so is $\mathcal{C}(X, Y)$. This theorem is generalized in Theorem 1, which states that if X is an $\mathbf{N}_0$ -space and Y is a paracompact $\mathbf{N}$ -space, then $\mathcal{C}(X, Y)$ is a paracompact $\mathbf{N}$ -space. It is unknown whether the preceding statement is true when "paracompact" is deleted.

DEFINITION 1. A k -network $\mathcal{O}$ for a space X is a family of subsets of X such that if $C \subset U$, with C compact and U open in X , then there is a finite union R of members of $\mathcal{O}$ such that $C \subset R \subset U$. A network $\mathcal{O}$ for a space X is a family of subsets of X such that if $x \in U$, with U open, then there is a $P \in \mathcal{O}$ such that $x \in P \subset U$.

In the preceding definition of k -network, if the finite union R is replaced by a member $P \in \mathcal{O}$, then $\mathcal{O}$ is called a *pseudobase* for X . Regular T_1 -spaces with countable pseudobases (i.e., $\mathbf{N}_0$ -spaces) and spaces with σ -locally finite networks (i.e., σ -spaces) have been investigated by E. Michael and A. Okuyama in [3] and [5], respectively. Though a k -network may not be a pseudobase, a space with a countable k -network also has a countable pseudobase.

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DEFINITION 2. An $\mathbf{N}$ -space is a regular T_1 -space with a σ -locally finite k -network.

The class of $\mathbf{N}$ -spaces clearly contains all metric spaces and all $\mathbf{N}_0$ -spaces. Moreover, each subspace of a (paracompact) $\mathbf{N}$ -space is a (paracompact) $\mathbf{N}$ -space¹ (just as each subspace of an $\mathbf{N}_0$ -space is an $\mathbf{N}_0$ -space).

The investigation of the properties of $\mathbf{N}$ -spaces has led to the problem of determining whether $\mathcal{C}(X, Y)$ is an $\mathbf{N}$ -space whenever X is an $\mathbf{N}_0$ -space and Y is an $\mathbf{N}$ -space. The space $\mathcal{C}(S, I)$, where S is the closed unit interval with discrete topology and I is the usual unit interval, is compact but not metrizable. It is known [8] that an $\mathbf{N}$ -space which is either first countable or locally compact is metrizable. It is therefore clear that some restriction of the sort made above must be placed on the space X , if $\mathcal{C}(X, Y)$ is to be an $\mathbf{N}$ -space whenever Y is so.

Here, we settle the question for paracompact $\mathbf{N}$ -spaces and obtain the added result that $\mathcal{C}(X, Y)$ is also paracompact. If $A \subset X$ and $B \subset Y$, then

$$W(A, B) = \{f: f \in \mathcal{C}(X, Y) \text{ and } f(A) \subset B\}.$$

THEOREM 1. If Y is a paracompact $\mathbf{N}$ -space and X is an $\mathbf{N}_0$ -space, then $\mathcal{C}(X, Y)$ is a paracompact $\mathbf{N}$ -space.

The proof is accomplished by establishing the following lemmas, the first of which incorporates the notion of a compact-covering map (map = continuous function). A map $f: S \rightarrow X$ is *compact-covering* if each compact subset of X is the image of a compact subset of S .

LEMMA 1. If $f: S \rightarrow X$ is a compact-covering map and Y is any space, then $\mathcal{C}(X, Y)$ is homeomorphic to a subspace of $\mathcal{C}(S, Y)$.

PROOF. It is easily shown that the map $\phi: \mathcal{C}(X, Y) \rightarrow \mathcal{C}(S, Y)$ defined by $\phi(g) = g \circ f$ is a homeomorphism onto its range.

Since each $\mathbf{N}_0$ -space is the compact-covering image of a separable metric space [3, Theorem 11.4(b)], Lemma 1, together with our earlier remark concerning the hereditary property of (paracompact) $\mathbf{N}$ -spaces, enables us to replace the space X of Theorem 1 by a separable metric space.

Let $\mathcal{S}$ be a subbase for a space X and let $\mathcal{O}$ be a σ -locally finite

¹ The regularity requirement in Definition 2 allows us to assume (as we do hereafter) that the members of a k -network for an $\mathbf{N}$ -space are closed sets. Since locally finite families have the closure preserving property and since an open subset of an $\mathbf{N}$ -space is the union of the members a k -network contained in it, each open subset of an $\mathbf{N}$ -space is an F_σ . The statement following Definition 2 now follows from [4, p. 836].

family of subsets of X such that if $C \subset U \in \mathcal{S}$, with C compact, then there is a finite union R of members of $\mathcal{O}$ such that $C \subset R \subset U$. We call such a family $\mathcal{O}$ (after Michael's $\mathcal{S}$ -pseudobase) an $\mathcal{S}$ - k -network. The following is almost verbatim the same as Michael's [3, Lemma 5.1].

LEMMA 2. *Let X be a Hausdorff space and let $\mathcal{S}$ be a subbase for X . Then X has a σ -locally finite k -network iff it has a σ -locally finite $\mathcal{S}$ - k -network.*

PROOF. The necessity of the condition is immediate. To prove the condition sufficient, suppose that $\mathcal{O} = \bigcup_{n \in \omega} \mathcal{O}_n$ is a σ -locally finite $\mathcal{S}$ - k -network for X . Let $\mathfrak{M}$ be the class of all finite subsets of ω and for each $E \in \mathfrak{M}$ let $\mathfrak{F}(E)$ be the class of all finite intersections of members of $\bigcup \{\mathcal{O}_n : n \in E\}$. Then each $\mathfrak{F}(E)$ is locally finite so that $\mathfrak{F} = \bigcup \{\mathfrak{F}(E) : E \in \mathfrak{M}\}$ is σ -locally finite. We shall show that $\mathfrak{F}$ is a k -network for X .

First suppose that $C \subset U \in \mathcal{B}$, where C is compact and $\mathcal{B}$ is a base for X consisting of all finite intersections of members of $\mathcal{S}$. Then $U = \bigcap_{i=1}^n \{S_i : S_i \in \mathcal{S}\}$ and, for each $i = 1, 2, \dots, n$, there is a finite union R_i of members of $\mathcal{O}$ such that $C \subset R_i \subset S_i$. Then $C \subset \bigcap_{i=1}^n R_i \subset U$ and $\bigcap_{i=1}^n R_i$ can be expressed as a finite union of members of $\mathfrak{F}$.

Now for an arbitrary open set, the proof proceeds exactly as in [3, Lemma 5.1].

It can be shown, using Lemma 2, that a countable product of $\mathbf{N}$ -spaces is an $\mathbf{N}$ -space and it follows from [6, Theorems 2.9 and 4.7] that if each factor is paracompact, then so is the product.

We shall need the following lemma which is again almost the same as one of Michael's results incorporated in the proof given for [3, Theorem 9.3].

LEMMA 3. *Let X be an $\mathbf{N}_0$ -space with countable pseudobase $\mathcal{O}$ and let Y be an $\mathbf{N}$ -space with σ -locally finite k -network $\mathfrak{J} = \bigcup_{n \in \omega} \mathfrak{J}_n$. Then for K compact in $\mathcal{C}(X, Y)$, C compact in X , and U open in Y with $K \subset W(C, U)$, there is a $P_n \in \mathcal{O}$ with $C \subset P_n$ and a finite union R of members of $\mathfrak{J}$ such that $K \subset W(P_n, R) \subset W(C, U)$.*

PROOF. We can assume (as we do hereafter) that the locally finite families $\mathfrak{J}_n$ (consisting of closed sets) are increasing, i.e., that $\mathfrak{J}_n \subset \mathfrak{J}_{n+1}$ for each $n \in \omega$ and that $\mathcal{O}$ is closed under the formation of finite intersections and finite unions. Let $\mathfrak{K}_n$ be the family of all finite subfamilies of $\mathfrak{J}_n$. For $F \subset \mathcal{C}(X, Y)$ and $A \subset X$, let $F(A) = \{f(x) : f \in F \text{ and } x \in A\}$. Now $K(C) \subset U$ and since the evaluation map $(f, x) \rightarrow f(x)$ from $\mathcal{C}(X, Y) \times C$ to Y is continuous (see, for instance, [1, 2.4, p. 260])

or [3, Lemma 9.1]), $K(C)$ is compact. For $H \subset \mathcal{C}(X, Y)$, $A \subset X$ and $\mathfrak{B} = \{B_1, B_2, \dots, B_k\}$, where $\mathfrak{B}$ is a finite family of subsets of Y , let $H(A) < \mathfrak{B}$ mean that $H(A) \subset \bigcup \mathfrak{B}$ and $H(A) \cap B_i \neq \emptyset$ for $i = 1, 2, \dots, k$. Then $\{\mathfrak{B} \mid \mathfrak{B} \in \mathfrak{R}_n, K(C) < \mathfrak{B}, \text{ and } \bigcup \mathfrak{B} \subset U\}$ is finite, since $K(C)$ has a neighborhood which meets only finitely many $T \in \mathfrak{I}_n$ for each $n \in \omega$. Let $\{\mathfrak{B}_n\}$ be an enumeration of those members of $\bigcup_{n \in \omega} \mathfrak{R}_n$ such that $K(C) < \mathfrak{B}_n$ and $\bigcup \mathfrak{B}_n \subset U$ for each n , and let $\mathfrak{B}'_m = \bigcup_{n=1}^m \mathfrak{B}_n$. Let $R'_m = \bigcup \mathfrak{B}'_m$. Now $V = \{x : K(x) \subset U\}$ is open in X [3, Lemma 9.2] and $C \subset V$. Let $\{P_n\}$ be an enumeration of those members of $\mathcal{P}$ such that $C \subset P_n \subset V$ for each n , and let $P'_m = \bigcap_{n=1}^m P_n$. We shall show that $K(P'_m) \subset R'_m$ for some m .

Suppose that $K(P'_m) \not\subset R'_m$ for each m . Then for each m there is an $x_m \in P'_m$ such that $K(x_m) \not\subset R'_m$. $A = C \cup \{x_1, x_2, \dots\}$ is compact [3, Lemma 8.1] and since $A \subset V$, $K(A)$ is compact and $K(A) \subset U$. Thus $K(A) \subset \bigcup \mathfrak{B} \subset U$ for some $\mathfrak{B} \in \bigcup_{n \in \omega} \mathfrak{R}_n$. Let $R = \bigcup_{i=1}^k T_i$ where $\{T_i\}_{i=1}^k = \mathfrak{B} \in \mathfrak{R}_n$ for some $n \in \omega$. Let $F = \bigcup \{T_i : T_i \cap K(C) = \emptyset\}$ and let $W = U - F$. Then W is open and $K(C) \subset W \subset U$. Now $V' = \{x : K(x) \subset W\} \cap V$ is open and $C \subset V'$. Thus there is an m such that $C \subset P'_m \subset V'$ and consequently, $K(P'_m) \subset W$. Then $K(x_m) \subset W \cap R \subset R - F \subset R'_n$ for some n . If $n \geq m$, then $P'_n \subset P'_m$ and $K(x_n) \subset R'_n$ and if $n \leq m$, then $R'_n \subset R'_m$ so that $K(x_m) \subset R'_m$. In either case we have a contradiction to the choice of the x_m 's. Thus $K(P'_m) \subset R'_m$ for some m and hence $K \subset W(P'_m, R'_m) \subset W(C, U)$.

LEMMA 4. *If Y is a paracompact $\aleph$ -space and X is a separable metric space, then $\mathcal{C}(X, Y)$ is an $\aleph$ -space.*

PROOF. Let $\{P_1, P_2, \dots\}$ be a countable k -network for X (closed under finite intersections) consisting of open sets, and let $\mathfrak{I} = \bigcup_{n \in \omega} \mathfrak{I}_n$ be a σ -locally finite k -network for Y (consisting of closed sets). For each $n \in \omega$, let $\mathfrak{B}_n$ be a locally finite open cover of Y such that each $B \in \mathfrak{B}_n$ meets only finitely many $T \in \mathfrak{I}_n$. Let $\mathfrak{R}_n$ be the family of all finite unions of members of $\mathfrak{I}_n$. (Here, we assume that $\mathfrak{I}_n \subset \mathfrak{I}_{n+1}$ for each n .)

Now for each pair $(n, m) \in \omega \times \omega$, let

$$\mathfrak{F}_{n,m} = \{W(P_m, B \cap R) : B \in \mathfrak{B}_n, R \in \mathfrak{R}_n\}.$$

Each $\mathfrak{F}_{n,m}$ is locally finite, for if $f \in \mathcal{C}(X, Y)$, $x_m \in P_m$ and U is a neighborhood of $f(x_m)$ which meets only finitely many $B \in \mathfrak{B}_n$, then $V = W(x_m, U)$ is a neighborhood of f which meets $W(P_m, B \cap R) \in \mathfrak{F}_{n,m}$ only if B meets U , and since each $B \in \mathfrak{B}_n$ meets only finitely many $T \in \mathfrak{I}_n$, V meets only finitely many members of $\mathfrak{F}_{n,m}$.

We shall show that the $(\sigma$ -locally finite) family of all finite inter-

sections of members of $\mathfrak{F} = \bigcup \{ \mathfrak{F}_{n,m} : (n, m) \in \omega \times \omega \}$ is an $\mathfrak{s}$ - k -network for $\mathcal{C}(X, Y)$, where $\mathfrak{s}$ is the subbase consisting of all sets of the form $W(C, U)$, with C compact in X and U open in Y .

Let $K \subset W(C, U) \in \mathfrak{s}$, with K compact in $\mathcal{C}(X, Y)$. Then, by Lemma 3, there is a P_m and an $R \in \mathcal{R}_n$ for some n , such that $C \subset P_m$ and $K(P_m) \subset R \subset U$. Let $\{B_i\}_{i=1}^t \subset \mathcal{B}_n$ be a cover of $K(C)$. Since K is evenly continuous on compacta [2, p. 236] and since Y is regular, for each $x \in C$, for each $y \in K(C)$ and for each $B_i(y)$ such that $y \in B_i(y)$, there is a neighborhood $V(x)$ of x (in C) and a closed neighborhood $W(y)$ of y (in $K(C)$) such that $f(V(x)) \subset B_i(y)$ whenever $f(x) \in W(y)$ for all $f \in K$. Let $\{W_j\}_{j=1}^s$ be a cover of $K(C)$ (where $W_j = W(y_j)$ for some $y_j \in K(C)$) and let $\{V_k\}_{k=1}^p$ be a cover of C (where $V_k = V(x_k)$ for some $x_k \in C$). Now there are compact subsets C_k such that $C = \bigcup_{k=1}^p C_k$ and $C_k \subset V_k$, $k = 1, 2, \dots, p$ [1, p. 152, 6.1]. Since each W_j is closed, each $H_{kj} = \{f \in K : f(x_k) \in W_j\}$ is compact and K is the union of the H_{kj} 's. Now for each k , $H_{kj} \subset W(C_k, B_i)$ for some i . Thus, as before, there is a P_k such that $C_k \subset P_k$ and $H_{kj}(P_k) \subset B_i$. Since $\{P_1, P_2, \dots\}$ is closed under finite intersections, we may assume that each $P_k \subset P_m$. Thus for each k , $H_{kj}(P_k) \subset B_i \cap R$ for some i . Then

$$K \subset \bigcap_{k=1}^p \left(\bigcup_{i=1}^t W(P_k, B_i \cap R) \right) \subset W(C, U).$$

Finally, $\bigcap_{k=1}^p (\bigcup_{i=1}^t W(P_k, B_i \cap R))$ can be expressed as a finite union of finite intersections of members of $\mathfrak{F}$, and so by Lemma 2, $\mathcal{C}(X, Y)$ is an $\aleph$ -space.

Now in order to complete the proof of Theorem 1, we prove the following highly specialized lemma and then show that the space $\mathcal{C}(X, Y)$ of Lemma 4 satisfies the conditions of this lemma.

First let us call a collection $\mathfrak{s}$ of subsets of a space Z a *subnetwork* for Z iff the family of all finite intersections of members of $\mathfrak{s}$ is a network for Z .

LEMMA 5. *Let $\mathfrak{s} = \bigcup_{n \in \omega} \mathfrak{s}_n$ be a subnetwork for a regular T_1 -space Z and suppose that for each $n \in \omega$, there is a locally finite open cover $\mathcal{V}_n$ of Z each of whose members meets only finitely many members of $\mathfrak{s}_n$. Then Z is a paracompact σ -space.*

PROOF. We may assume that $\mathfrak{s}_n \subset \mathfrak{s}_{n+1}$ for each $n \in \omega$. The condition on the $\mathcal{V}_n$'s implies that $\mathfrak{s}$ is σ -locally finite and consequently, so is the family $\mathfrak{I}$ of all finite intersections of members of $\mathfrak{s}$. Thus Z is a σ -space. Let $\mathfrak{I}_n$ be the family of all finite intersections of members of $\mathfrak{s}_n$ so that $\mathfrak{I} = \bigcup_{n \in \omega} \mathfrak{I}_n$ and each $V \in \mathcal{V}_n$ meets only finitely many

$T \in \mathfrak{J}_n$. For $T \in \mathfrak{J}_n$, let $W(T, n) = \bigcup \{V \in \mathfrak{V}_n : V \cap T \neq \emptyset\}$ (sometimes called the *star* of T with respect to $\mathfrak{V}_n$ and written $\text{St}(T, \mathfrak{V}_n)$). Now $T \subset W(T, n)$ and $W(T, n)$ is open. For $z \in Z$,

$$U(z) = Z - \bigcup \{\text{Cl}(V) : z \notin \text{Cl}(V), V \in \mathfrak{V}_n\}$$

is a neighborhood of z which meets $W(T, n)$ only if there is a $V \in \mathfrak{V}_n$ and a $T \in \mathfrak{J}_n$ such that $V \cap T \neq \emptyset$ and $z \in \text{Cl}(V)$. Since $z \in \text{Cl}(V)$ for only finitely many $V \in \mathfrak{V}_n$ and since $V \cap T \neq \emptyset$ for only finitely many $T \in \mathfrak{J}_n$ for each $V \in \mathfrak{V}_n$, $\mathfrak{W}_n = \{W(T, n) : T \in \mathfrak{J}_n\}$ is locally finite.

Now let $\mathfrak{U}$ be an open cover of Z for each $n \in \omega$; let

$$\mathfrak{J}_n(\mathfrak{U}) = \{T \in \mathfrak{J}_n : T \subset U_T \text{ for some } U_T \in \mathfrak{U}\}.$$

For each $n \in \omega$ and for each $T \in \mathfrak{J}_n(\mathfrak{U})$, $T \subset W(T, n) \cap U_T = W'(T)$ and the family $\mathfrak{W}'_n = \{W'(T) : T \in \mathfrak{J}_n(\mathfrak{U})\}$ is locally finite. Then $\bigcup_{n \in \omega} \mathfrak{W}'_n$ is a σ -locally finite open refinement of $\mathfrak{U}$ and hence Z is paracompact.

LEMMA 6. *The space $\mathcal{C}(X, Y)$ of Lemma 4 satisfies the conditions of Lemma 5.*

PROOF. Let D be a countable dense subset of X and let $\mathfrak{D}$ be the collection of all nonempty finite subsets of D . Let $\{P_n\}$ be a countable pseudobase for X consisting of open sets and let $\mathfrak{J} = \bigcup_{n \in \omega} \mathfrak{J}_n$ be a σ -locally finite k -network for Y consisting of closed sets, again assuming that $\mathfrak{J}_n \subset \mathfrak{J}_{n+1}$ for each $n \in \omega$. For a finite collection $\mathfrak{R}$ of subsets of Y and for $E \subset X$ and $A \subset X$, let

$$\begin{aligned} [A, \mathfrak{R}, E] \\ = \{f \in \mathcal{C}(X, Y) : f(A) \subset \bigcup \mathfrak{R}, f(E) \cap T \neq \emptyset \text{ for each } T \in \mathfrak{R}\}. \end{aligned}$$

Then for $(n, m) \in \omega \times \omega$ and for $E \in \mathfrak{D}$, let

$$\mathfrak{F}(n, m, E) = \{[P_n, \mathfrak{R}, E] : \mathfrak{R} \subset \mathfrak{J}_m\}$$

and let

$$\mathfrak{F} = \bigcup \{\mathfrak{F}(n, m, E) : (n, m, E) \in \omega \times \omega \times \mathfrak{D}\}.$$

We shall show that $\mathfrak{F}$ is a subnetwork for $\mathcal{C}(X, Y)$ satisfying the conditions of Lemma 5.

To show that $\mathfrak{F}$ is a subnetwork, we need only show that if $f \in W(C, U)$, with C compact in X and U open in Y , then there is an $F \in \mathfrak{F}$ such that $f \in F \subset W(C, U)$. Now if $f \in W(C, U)$, then by Lemma 3, there is a P_n and a finite union $\bigcup_{i=1}^k T_i$ of members of $\mathfrak{J}_m$, for some m , such that $f \in W(P_n, \bigcup_{i=1}^k T_i) \subset W(C, U)$. We may assume that $\mathfrak{R} = \{T_1, T_2, \dots, T_k\}$ is an irreducible cover of $f(P_n)$ [1, p. 160, 1.1]. Since each T_i is closed,

$$F_i = \bigcup \{T_j : j \neq i \text{ and } 1 \leq j \leq k\}$$

is closed and since $\mathfrak{R}$ is irreducible, we have that $(Y - F_i) \cap f(P_n)$ is nonempty and is contained in T_i . Thus for $1 \leq i \leq k$, there is an $x_i \in P_n \cap D$ such that $f(x_i) \in T_i$. Then, for $E = \{x_1, x_2, \dots, x_k\} \in \mathfrak{D}$, $f \in [P_n, \mathfrak{R}, E] \subset W(C, U)$.

Now it suffices to show that, for $(n, m, E) \in \omega \times \omega \times \mathfrak{D}$, each $f \in \mathfrak{C}(X, Y)$ has a neighborhood which meets only finitely many members of $\mathfrak{T}(n, m, E)$ and that the collection of all these neighborhoods is locally finite. Fix n, m , and E and let $U(f)$ be an open neighborhood of $f(E)$ which meets only finitely many $T \in \mathfrak{T}_m$. $\mathfrak{U} = \{U(f) : f \in \mathfrak{C}(X, Y)\}$ covers Y and so $\mathfrak{U}$ has a locally finite open refinement $\mathfrak{V}$. Let $\mathfrak{V}_f$ be a finite subcollection of $\mathfrak{V}$ such that $f(E) \subset \bigcup \mathfrak{V}_f$ and $f(E) \cap V \neq \emptyset$ for each $V \in \mathfrak{V}_f$. Then

$$\langle E, \mathfrak{V}_f \rangle = \{g : g(E) \subset \bigcup \mathfrak{V}_f \text{ and } g(E) \cap V \neq \emptyset \text{ for each } V \in \mathfrak{V}_f\}$$

is a neighborhood of f which meets only finitely many members of $\mathfrak{T}(n, m, E)$. Finally, for $f \in \mathfrak{C}(X, Y)$, $f(E)$ has a neighborhood U which meets only finitely many $V \in \mathfrak{V}$. Then $W(E, U)$ is a neighborhood of f which meets only finitely many members of

$$\{\langle E, \mathfrak{V}_g \rangle : g \in \mathfrak{C}(X, Y)\}.$$

That completes the proof.

It should be noted that the first part of the proof of Lemma 6 implies that, if Y is an $\mathfrak{N}$ -space (not necessarily paracompact) and X is separable metric, then $\mathfrak{C}(X, Y)$ is a σ -space. For in that part of the proof, no use was made of the paracompact character of Y . Also as a corollary to Theorem 1, we have that $\mathfrak{C}(X, Y)$ is paracompact whenever X is separable metric and Y is metric.

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