PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 29, Number 1, June 1971

ITERATIVE SOLUTION OF A WIENER-HOPF PROBLEM IN SEVERAL VARIABLES[1]

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ABSTRACT. An extension of the classical problem of Wiener and Hopf to functions of several complex variables is considered. A sufficient condition for the unique solvability of the problem is obtained. Finally, a method for an iterative construction of the solution is given.

1. Introduction. A recent analysis of a diffraction problem has led Kraut and Lehman [1] to extend a classical problem of Wiener and Hopf [2], [3] to functions of several complex variables. To pose the problem let R^n be the real n-space and C^n the space of n complex variables $z=(z_1, \cdots, z_n)$, $z_j=x_j+iy_j$. Let $S_0 \subset R^n$ be the set $\{y=(y_1, \cdots, y_n): \gamma_j<y_j<\delta_j\}$, and by the *tube* T_0 with the *basis* S_0 denote the set $\{z=x+iy \in C^n: y \in S_0\}$. To each of the 2^n possible choices of $y_j>\gamma_j$ or $y_j<\delta_j$ attach basis sets S_p and corresponding tubes T_p, $p=1, \cdots, 2^n$, in some order, such that $S_1=\{y: \gamma_j<y_j, j=1, \cdots, n\}$. By $A(T_p)$ denote the class of functions $f: T_p \to C$ which are analytic on T_p, and by $A_0(T_p)$ denote all $f \in A(T_p)$, $p \neq 0$, obeying $f(z) \to 0$ as any $|y_j| \to \infty$ in S_p. Finally, by $H(T_p)$ denote the class of all $f \in A(T_p)$ such that

$$\|f\|_p = \sup_{y \in S_p} \left\{ \int \overset{+\infty}{\underset{-\infty}{\cdots}} \int |f(z)|^2 dx_1 \cdots dx_n \right\}^{1/2} < \infty. \tag{1}$$

The extended Weiner-Hopf problem (henceforth abbreviated to the EWH problem) can be stated as follows: *Given a function $k \in A(T_0)$ and bounded on T_0 and given $g_1 \in H(T_1)$, find 2^n functions $f_p \in H(T_p)$, $p=1, \cdots, 2^n$, such that*

$$k(z)f_1(z) = g_1(z) + \sum_{p=2}^{2^n} f_p(z), \qquad \forall z \in T_0. \tag{2}$$

2. The main results. This section is devoted to the proof of Theorems 1 and 2 below.

Presented to the Society, January 22, 1970 under the title *A Wiener-Hopf problem in several variables*; received by the editors January 12, 1970.

AMS 1969 *subject classifications.* Primary 3215, 4240; Secondary 4515, 7845.

Key words and phrases. Multidimensional Wiener-Hopf problem, multiple Fourier transforms, tubular domains, electromagnetic diffraction, contraction mapping.

[1] This research was supported in part by National Science Foundation grant GP-12407.

THEOREM 1. *If the range of k for $z \in T_0$ lies in a closed, bounded, convex set which does not contain the origin, then the EWH problem has a unique solution.*

It is noted that under the conditions of Theorem 1, $\exists$ a closed disk in C which contains the range of k and does not contain the origin. Let $w \neq 0$ be the center of this disk, and note that its radius is less than $|w|$. Thus, if

$$h(z) = k(z) - w, \qquad z \in T_0, \tag{3}$$

it follows that $h(z) \in A(T_0)$, is bounded on T_0, and

$$\sup_{z \in T_0} (|h(z)|) < |w|. \tag{4}$$

Next, introduce the notation $s(T_p)$ to denote the *spine* of T_p, $p \neq 0$. That is, if S_p is given by inequalities of the forms $\gamma_j < y_j$ or $y_j < \delta_j$, denoting by ∂S_p^j the lines $(-\infty + i\gamma_j, \infty + i\gamma_j)$ or $(\infty + i\delta_j, -\infty + i\delta_j)$, respectively, then $s(T_p) = \prod_{j=1}^{n} \partial S_p^j$.

Finally, if $f \in H(T_p)$, $p \neq 0$, note that (Bochner [4]) the Cauchy integral formula

$$f(z) = (2\pi i)^{-n} \int \cdots \int_{s(T_p)} f(\zeta) \prod_{j=1}^{n} (\zeta_j - z_j)^{-1} d\zeta_j, \quad z \in T_p,$$

is valid, where the boundary value $f(\zeta)$ of $f(z)$ is square integrable over R^n as a function of x, and is attained a.e. as $z \to \zeta \in s(T_p)$, $z \in T_p$.

THEOREM 2. *Let k satisfy the conditions of Theorem 1, and let $w \in C$ and $h(z)$ be given as in (3). Define the operator $L: H(T_0) \to H(T_1)$ by*

$$L(f)(z) = w^{-1}\left[g_1(z) - (2\pi i)^{-n} \int \cdots \int_{s(T_1)} h(\zeta) f(\zeta) \prod_{j=1}^{n} (\zeta_j - z_j)^{-1} d\zeta_j \right], \quad f \in H(T_0). \tag{5}$$

Then the sequence f_1^j given by

$$\overset{0}{f_1} = g_1, \qquad \overset{j}{f_1} = L[\overset{j-1}{f_1}], \tag{6}$$

converges uniformly on compact subsets of T_1 to the f_1 of the solution $\{f_p\}_{p=1}^{2^n}$ of the EWH problem. The remaining f_p, $p \neq 1$, are given by

$$f_p(z) = (2\pi i)^{-n} \int \cdots \int_{s(T_p)} [k(\zeta) f_1(\zeta) - g_1(\zeta)] \prod_{j=1}^{n} (\zeta_j - z_j)^{-1} d\zeta_j, \quad z \in T_p. \tag{7}$$

Before proceeding with the proofs of these theorems, we present five lemmas, some of which are in the literature, but are listed here for ease of future reference. The first is a special case of a theorem of Bochner [5].

LEMMA 1. *$F\in H(T_p)$ for some p, iff $\exists$ a measurable function $\phi(t)=\phi(t_1, \cdots, t_n)$ on R^n such that*

$$(8) \qquad \phi_y(t) = \phi(t)\exp\left(-\sum_{j=1}^{n} y_j t_j\right) \in L^2(R^n), \qquad \forall y \in S_p,$$

(i.e., $\phi_y(t)$ is square integrable) and

$$(9) \qquad F(z) = \int_{-\infty}^{\infty}\cdots\int \exp\left(\sum_{j=1}^{n} iz_j t_j\right)\phi(t)dt_1\cdots dt_n, \qquad z\in T_p.$$

Moreover,

$$(10) \qquad \|F\|_p = \sup_{y\in S_p}\left((2\pi)^{n/2}\int_{-\infty}^{\infty}\cdots\int |\phi_y(t)|^2 dt_1\cdots dt_n\right)^{1/2}.$$

For the proof see [5] or else Bochner and Martin [6]. The next result is essentially also due to Bochner [5].

LEMMA 2. *If $F\in H(T_0)$, $\exists$ 1 set $\{F_p\}_{p=1}^{2^n}$ of functions $F_p\in A_0(T_p)\ni$*

$$(11) \qquad F(z) = \sum_{p=1}^{2^n} F_p(z), \qquad z\in T_0,$$

and

$$(12) \qquad F_p(z) = L_p[F](z) = (2\pi i)^{-n}\int_{s(T_p)}\cdots\int F(\zeta)\prod_{j=1}^{n}(\zeta_j - z_j)^{-1}d\zeta_j, \qquad z\in T_p.$$

For the proof see Bochner [5], and for the uniqueness part of the result see Kraut et al. [7].

LEMMA 3. *If $f_p\in H(T_p)$ for some $p=1, \cdots, 2^n$, then $f_p\in A_0(T_p)$. Moreover, if $f_p^j\in H(T_p)$, $j=1, 2, \cdots$, and if $\|f_p^j - f_p\|_p\to 0$, as $j\to\infty$, then $f_p^j\to f_p(z)$ uniformly on compact subsets of T_p.*

PROOF. If $z^0\in T_p$ and if $l_p\in H(T_p)$, then l_p can be expressed as a convergent Taylor series in a polydisk K with center z^0 and nonzero radius $r=(r_1, \cdots, r_n)$. Moreover, $r_j = y_j^0 - \gamma_j$ or $r_j = \delta_j - y_j^0$, are the greatest possible radii of K. Thus

$$\int \cdots \int_K |l_p(z)|^2 \prod_{j=1}^n dx_j dy_j$$

$$(13) \quad \begin{aligned} &= \int_0^{2\pi} \cdots \int \int_0^{r_1} \cdots \int_0^{r_n} \left| \sum_{|\alpha|=0}^{\infty} a_\alpha \exp\left(i \sum_{j=1}^n \alpha_j \theta_j\right) \prod_{j=1}^n r_j^{\alpha_j} \right|^2 \prod_{j=1}^n r_j dr_j d\theta_j \\ &= (2\pi)^n \sum_{|\alpha|} |a_\alpha|^2 \prod_{j=1}^n r_j^{2\alpha_j+2} (2\alpha_j + 2)^{-1} \\ &\geqq \prod_{j=1}^n \pi r_j^2 |l_p(z^0)|^2, \end{aligned}$$

where $\alpha = (\alpha_1, \cdots, \alpha_n)$, $|\alpha| = \sum_{j=1}^n \alpha_j$; and $a_0 = l(z^0)$ was used in the last step. Now

$$(14) \qquad \int \cdots \int_K |l_p(z)|^2 dx_1 dy_1 \cdots dx_n dy_n \leqq 2^n \|l_p\|_p^2 \prod_{j=1}^n r_j.$$

Combining (13) and (14) it is seen that

$$(15) \qquad |l_p(z^0)| \leqq (2\pi^{-1})^{n/2} \prod_{j=1}^n r_j^{-1/2} \|l_p\|_p.$$

Hence, as any $|y_j| \to \infty$ in T_p it is seen that $r_j \to \infty$ and $l_p(z^0) \to 0 \Rightarrow l_p \in A_0(T_p)$, and the first part of the lemma is established.

Next, if U is a compact subset of T_p, then the distance from U to ∂T_p is positive; hence, $\exists \beta_j \ni 0 < \beta_j < r_j$, $j = 1, \cdots, n$, holds $\forall z^0 \in U$. Thus

$$(16) \qquad |l_p(z^0)| \leqq (2\pi^{-1})^{n/2} \prod_{j=1}^n \beta_j^{-1/2} \|l_p\|_p, \qquad \forall z^0 \in U,$$

and convergence in the $\|\cdot\|_p$ norm implies uniform convergence in U. This completes the proof of the lemma.

LEMMA 4. *If* $F \in H(T_0)$, $\exists$ 1 *set* $\{F_p\} = \{L_p[F]\}$, $p = 1, \cdots, 2^n$, *of functions* $F_p \in H(T_p) \ni$

$$(17) \qquad F(z) = \sum_{p=1}^{2^n} F_p(z) = \sum_{p=1}^{2^n} L_p[F](z), \qquad z \in T_0,$$

where the L_p *are defined by* (12), *and*

$$(18) \qquad \|L_p[F]\|_0 \leqq \|F\|_0, \qquad p = 1, \cdots, 2^n.$$

PROOF. Only the existence of the $F_p \in H(T_p)$, and hence $F_p \in A_0(T_p)$ by Lemma 3, satisfying the first part of (17) and (18) need be shown, since uniqueness, and thence, the remainder of (17) will follow from Lemma 2.

From Lemma 1 $\exists$ $\phi(t)$ measurable in $R^n \ni$ (8) and (9) hold with $p=0$. The integral in (9) may be written as the sum of 2^n integrals obtained by writing each of the single integrals as the sum of an integral on $(-\infty, 0)$ and one on $(0, \infty)$. With each S_p associate one of these 2^n integrals, denoted by $(\int \cdots \int)_p$, as follows: whenever $y_j > \gamma_j$ $(y_j < \delta_j)$ enters in the definition of S_p, the jth variable is integrated on $(0, \infty)$ $((-\infty, 0))$. Then

$$F(z) = \sum_{p=1}^{2^n} \left(\int \cdots \int \right)_p \exp\left(\sum_{j=1}^{n} i z_j t_j \right) \phi(t) dt_1 \cdots dt_n, \qquad z \in T_0. \tag{19}$$

It will now be shown that the first integral $(p=1)$ yields the F_1 in (17).

From (8) and Fubini's theorem $\int_{-\infty}^{\infty} \exp(-2y_1 t_1) |\phi(t)|^2 dt_1$ exists for $\gamma_1 < y_1 < \delta_1$ for almost all $t' = (t_2, \cdots, t_n) \in R^{n-1}$. Hence, $\int_0^\infty \exp(-2y_1 t_1) |\phi(t)|^2 dt_1$ converges for $\gamma_1 < y_1$, and

$$\exp\left(- \sum_{j=2}^{n} 2 y_j t_j \right) \int_0^\infty \exp(-2 y_1 t_1) \, | \phi(t) |^2 dt_1$$

is integrable for $t' \in R^{n-1}$ and for $\gamma_j < y_j < \delta_j$, $j = 2, \cdots, n$. Repeating the above argument for the remaining $(n-1)$ variables, it follows that $(\int \cdots \int)_1 (-\sum_{j=1}^n 2 y_j t_j) |\phi(t)|^2 \prod_{m=1}^n dt_m$ converges for $y \in S_1$. Defining $\phi(t)$ to be zero when $y_j < 0$ and using Lemma 1, it is seen that the first term in (19) belongs to $H(T_1)$, and may be set equal to F_1. Applying the same argument to the remaining integrals for $p = 2, \cdots, 2^n$, in (19), the first part of (17) is established.

From (10) of Lemma 1 it is seen that

$$\begin{aligned} \|F_p\|_0^2 &= \sup_{y \in S_0} (2\pi)^{n/2} \left(\int \cdots \int \right)_p | \phi_y(t) |^2 dt_1 \cdots dt_n \\ &\leq \sup_{y \in S_0} (2\pi)^{n/2} \int_{-\infty}^{\infty} \cdots \int | \phi_y(t) |^2 dt_1 \cdots dt_n = \|F\|_0^2. \end{aligned} \tag{20}$$

This completes the proof of Lemma 4.

LEMMA 5. *If* $f_1 \in H(T_1)$ *then*

$$\|f_1\|_1 = \|f_1\|_0 = \lim_{y \to \gamma} \left\{ \int_{-\infty}^{\infty} \cdots \int |f(z)|^2 dx_1 \cdots dx_n \right\}^{1/2}. \tag{21}$$

PROOF. Let

$$M_1(y;f) = \left\{ \int_{-\infty}^{\infty} |f(z)|^2 dx_1 \right\}^{1/2},$$

$$M(y;f) = \left\{ \int_{-\infty}^{\infty} \cdots \int |f(z)|^2 dx_1 \cdots dx_n \right\}^{1/2}.$$

It is known (see Hille [8]) that $M_1(y;f)$ is a nonincreasing function of y_1. Now, if $b_1 < y_1$,

$$\int_{-\infty}^{\infty} \cdots \int [M_1(y;f)]^2 dx_2 \cdots dx_n \leq \int_{-\infty}^{\infty} \cdots \int [M_1(b_1, y_2, \cdots, y_n; f)]^2 dx_2 \cdots dx_n.$$

Thus, $M(y;f) \leq M(b_1, y_2, \cdots, y_n, f)$.

In a similar way, it can be shown that $M(y;f)$ is a nonincreasing function of y_j, $j = 2, \cdots, n$. This fact and the definition (1) of $\|\cdot\|_0$ and $\|\cdot\|_1$ establish the lemma.

PROOF OF THEOREM 1. It is first noted that $H(T_0)$ is a linear space with scalars in C and that $\|\cdot\|_0$ is a norm over this space. It follows readily, using Lemma 3 (see [6, Chapter VI] for example) that $H(T_0)$ is a Banach space.

Consider the operator defined by (5) and use the notation in (12) to obtain

$$L[f] = w^{-1}(g_1 - L_1[hf]), \qquad f \in H(T_0). \tag{22}$$

Since h is bounded and $h \in A(T_0)$, $hf \in H(T_0)$ and, by Lemma 4, $L_1[hf] \in H(T_1)$. Thus L is a linear operator.

$$L: H(T_0) \to H(T_1) \subset H(T_0). \tag{23}$$

From (22), the linearity of L_1 and Lemma 4, it is seen that if F and $G \in H(T_0)$,

$$\|L[F] - L[G]\|_0 = |w|^{-1} \|L_1[h(F-G)]\|_0 \leq |w|^{-1} \|h(F-G)\|_0. \tag{24}$$

Now, from (1), (4) and (24),

$$\|L[F] - L[G]\|_0 < |w|^{-1} \sup_{z \in T_0} |h(z)| \, \|F - G\|_0 < \alpha \|F - G\|_0, \tag{25}$$

where $0<\alpha<1$. Thus L is a contraction mapping and $\exists\ 1\ f\in H(T_0)$ $\ni L[f]=f$. But, from (23), it follows that $f=f_1\in H(T_1)$, and hence $\exists\ 1 f_1\in H(T_1)\ \ni$

$$L[f_1] = f_1. \tag{26}$$

From (3), (5) and (26), it is seen that $wf_1=g_1-L_1[(k-w)f_1]$, whence, using (26) again, it follows that

$$L_1[kf_1] = g_1. \tag{27}$$

However, $kf_1\in H(T_0)$ and by Lemma 4 and (27), $\exists\ 1$ set $\{L_p[kf_1]\}_{p=1}^{2^n}$, $L_p[kf_1]\in H(T_p)\ \ni$

$$kf_1(z) = \sum_{p=1}^{2^n} L_p[kf_1](z) = g_1(z) + \sum_{p=2}^{2^n} L_p[kf_1](z), \qquad z \in T_0. \tag{28}$$

This completes the proof of Theorem 1.

Proof of Theorem 2. From the proof of Theorem 1, it follows that the sequence f_1^j, $j=0, 1, \cdots$, defined in (6) converges in the $\|\cdot\|_0$ norm to the f_1 of the solution $\{f_p\}_{p=1}^{2^n}$ of the EWH problem. Moreover, $f_1^j\in H_1(T_1)$ $\forall j$, and by Lemma 5, $f_1^j\to f_1$ in the $\|\cdot\|_1$ norm and hence uniformly in compact subsets of T_1. This establishes the first part of Theorem 2. To complete the proof, note that equation (7) is a consequence of equation (28) and Lemma 4.

3. **Remarks on previous results.** As a special case of Theorem 1 it is noted that if $k(z)$ obeys

$$|1 - k(z)| < 1, \qquad \forall z \in T_0, \tag{29}$$

then the EWH problem is uniquely solvable. Kraut [9] and Kraut and Lehman [1] considered the case $\gamma_j<0<\delta_j$, $j=1, \cdots, n$, and the class of kernels $k(z)=1-\lambda h(z)$, where λ is a real parameter which can be varied. They replace the condition (29) by the weaker condition $|1-k(z)|<1$, $z_j=x_j$, $j=1, \cdots, n$. This latter condition is not sufficient for the unique solvability of the EWH problem, as is seen by taking (for $n=2$), $k(z)=1+[2(z_1+i)(z_2+i)]^{-1}$, $g(z)=(z_1+i)^{-1}\cdot(z_2+i)^{-1}$, and $T_0=\{z\in C^2: -3/4<y_j<3/4\}$. The resulting EWH problem satisfies this latter condition but has no solution. However, since the $h(z)$ in [1] and [9] is uniformly bounded in T_0, λ can be chosen small enough for (29) to hold in T_0, and from Theorem 1 the resulting EWH problem has a unique solution.

The author was introduced to this problem by Dr. Edgar Kraut. He is indebted to him as well as to Professor Robert Borrelli and Dr. William Hall for a number of discussions and suggestions.

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