SOME SPACES THAT DO NOT HAVE THE COMMON FIXED POINT PROPERTY

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ABSTRACT. For what topological spaces X do every pair of self maps of X which commute under composition have a common fixed point? No nontrivial examples of such spaces are known. Since every self map commutes with itself, X does not have this property if X does not have the fixed point property. It is shown that every completely regular Hausdorff space containing an arc does not have this property. In general, the self maps for these spaces are not surjective. The image is the arc. For surjective self maps it is shown that every topological manifold with nonnegative Euler characteristic does not have this property. An earlier counterexample for the closed interval is used in all proofs. This counterexample is due to Huneke.

A topological space X is said to have property CFP² provided every two self maps which commute under composition have a common fixed point. In [4] Huneke shows that the closed unit interval, I , does not have CFP². However if one of the two self maps of this interval is open, Joichi and Folkman (see [6]) show that commuting self maps of I do have a common fixed point.

E. R. Fadell raised the question whether the fixed point property implies CFP² for simply connected closed topological manifolds. A first negative answer to this question is given by Proposition 1 which says that every topological manifold M does not have CFP² by virtue of self maps f, g which have the property that $f(M) \cup g(M) \subset A$, an arc in M .

As a result we study surjective maps. Let X have property CFP²S if X has CFP² for surjective self maps. The main result of this paper (Corollary 1) is that all topological manifolds M with nonnegative Euler characteristic, do not have CFP²S. Huneke [5] first obtained this result for projective spaces over the real and complex fields.

A natural question is whether all manifolds M fail to have CFP²S. For this it is sufficient to know that all manifolds with the fixed point property have positive Euler characteristic. This result is unknown to the writers.

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Another question is whether X has CFP² for homeomorphisms. The answer is unknown to the writers even for the unit disk.

Our technique of construction is to find a surjective self map $f: M \rightarrow M$, homotopic to the identity map, and with fixed point set whose components have invariant and complement invariant neighborhoods (see Theorem 1). We then use the counterexample to CFP² for the closed interval locally and in one coordinate to construct the desired maps.

PROPOSITION A. *Let X be a completely regular Hausdorff space which contains an arc. Then there exist two continuous maps from X onto the arc in X which commute without common fixed points.*

PROOF. In [4] it is shown that there exist two self maps of $[-1, 1]$, f and g , which commute without common fixed points. Let h be a homeomorphism from the arc in X onto $[-1, 1]$. Let $\bar{h}$ be Tietze's extension of h to βX , the Stone-Čech compactification of X . Then $h^{-1} \cdot f \cdot \bar{h}|_X$ and $h^{-1} \cdot g \cdot \bar{h}|_X$ are self maps of X which commute without common fixed points.

NOTATION. Throughout let R^m denote m -dimensional euclidean space, $m > 0$. Let $H^m = \{(x_1, \dots, x_m) \in R^m : x_m \geq 0\}$ denote m -dimensional euclidean *half space*. Let M denote a *topological manifold of dimension m* , i.e., a Hausdorff space such that each point has an open neighborhood which is either homeomorphic to R^m or else to H^m . If $x \in M$ has an open neighborhood homeomorphic to R^m we say that $x \in \text{interior } M$ ($\text{int } M$). If $x \notin \text{int } M$ then we say that $x \in \text{boundary } M$ ($\text{bd } M$).

If M is compact and $\text{bd } M = \emptyset$ we say that M is *closed*.

If M is not compact we say that M is *open*.

We have an immediate consequence of Proposition A.

COROLLARY A. *Every topological manifold M does not have CFP².*

NOTATION. Let M be a topological manifold. Let M_0 denote the complement of a point interior to M . Let χM denote the Euler characteristic of M .

LEMMA 1. *Every connected topological manifold M , with empty boundary, admits a surjective self map $f: M \rightarrow M$ such that*

- (1) f is ϵ -homotopic to the identity map.
- (2) f has at most one fixed point.
- (3) f is fixed point free if M is open or $\chi M = 0$.

PROOF. The pair of maps $M \xrightarrow{\Delta} M \times M \xrightarrow{p_1} M$ defined by $\Delta(x) = (x, x)$, $p_1(x, y) = x$ is a basic example of a microbundle. Kister and Mazur

(see [7]) have shown that every microbundle over a paracompact base space contains a fiber bundle with euclidean fiber. For this example this means that there exists an open neighborhood of $\Delta(M)$ in $M \times M$, TM , such that $\text{pr}_1|_{TM}$ is a fiber bundle with euclidean fiber. We call this bundle "the" tangent bundle of M and let T_0M denote $TM - \Delta(M)$. Since $\pi_i(R^m, R_0^m) = 0$, if $i < m$, and $H^m M_0 = 0$ we may use obstruction theory to deform the section Δ into a section $s: M \rightarrow TM$ such that $s(M_0) \subset T_0M$. As a result $f = \text{pr}_2 \cdot s$ is a map with exactly one fixed point. Here $\text{pr}_2(x, y) = y$. Let $d \in H^m M$ denote the obstruction to deforming s to a section $s_0: M \rightarrow TM$ such that $s(M) \subset T_0M$. If M is open, $H^m M = 0$, and if M is closed, R. Brown [2] shows that $d = X$, the Euler class of M . Recall that $X = \chi M \cdot \mu$, where μ is the fundamental cohomology class of M . As a result $f = \text{pr}_2 \cdot s_0$ is the map desired in (3) above. Note that since f is homotopic to the identity map it is surjective. Q.E.D.

LEMMA 2. *Every topological manifold M admits a surjective self map $f: M \rightarrow M$ such that*

- (1) *f is ϵ -homotopic to the identity map.*
- (2) *$f(\text{bd } M) \subset \text{bd } M$.*
- (3) *f has at most one interior fixed point and has at most one fixed point on each component C , of $\text{bd } M$.*
- (4) *f is fixed point free on $\text{int } M$ if M is open or $\chi M = 0$. f is fixed point free on C , a component of $\text{bd } M$, if $\chi C = 0$ or C is open.*

PROOF. Let $f: \text{bd } M \rightarrow \text{bd } M$ be the map given on each component of $\text{bd } M$ by Lemma 1. Morton Brown [1] has shown that $\text{bd } M$ has a neighborhood, N , homeomorphic to $\text{bd } M \times I$. As a result we can extend f to $\text{int } N$ with the desired properties. Define $s: N \rightarrow N \times N$ by $s(x) = (x, f(x))$. Since $H^m(M_0, \text{bd } M) = 0$, we may use obstruction theory to extend the deformation of the section Δ to $s: \text{int } M \rightarrow T(\text{int } M)$ such that $s(\text{int } M) \subset T_0(\text{int } M_0)$. As a result $f = \text{pr}_2 \cdot s$ is a surjective self map with properties (1)–(4).

NOTATION. Recall that $U \subset M$ is called *invariant* under $f: M \rightarrow M$ if $f(U) \subset U$. We say that $U \subset M$ is *complement invariant* under $f: M \rightarrow M$ if $f(M - U) \subset (M - U)$. Let $U \subset R^m$ be a euclidean neighborhood of the origin and let $f: U \rightarrow R^m$ be a map with exactly one fixed point at the origin. Recall that the *index* of this fixed point under f is defined to be the degree of the difference map $\delta f: U_0 \rightarrow R_0^m$ defined by $\delta f(x) = f(x) - x$. Here R_0^m and U_0 denote the complement of the origin in R^m and U respectively.

Let $rB^m = \{x \in R^m: |x| \leq r\}$ denote the closed ball in R^m of radius r .

LEMMA 3. *Let the origin be the only fixed point for $f: 3B^{2k} \rightarrow R^{2k}$, and let $f(3B^{2k} - 2B^{2k}) \cap B^{2k} = \emptyset$. Suppose the index of the fixed point is 1. Then there exists a map $g: 3B^{2k} \rightarrow R^{2k}$ such that*

- (1) $g|_{(3B^{2k} - 2B^{2k})} = f|_{(3B^{2k} - 2B^{2k})}$.
- (2) *The origin is the only fixed point under g .*
- (3) B^{2k} *is invariant and complement invariant under g .*
- (4) $g|_{B^{2k}}$ *is given by $g(x) = -x$.*

PROOF. Note that there exists a homotopy equivalence between B_0^{2k} and $3B^{2k} - 2B^{2k}$. Since the index of the fixed point under both f and g is 1, $sg|_{B_0^{2k}}: B_0^{2k} \rightarrow T_0^{2k} R^{2k}$ is homotopic to $sf|_{(3B^{2k} - 2B^{2k})}: 3B^{2k} - 2B^{2k} \rightarrow T_0 R^{2k}$ under this equivalence. As a result there exists an extension $s: 3B^{2k} \rightarrow TR^{2k}$ such that $s|_{B^{2k}} = sg|_{B^{2k}}$, $s|_{3B^{2k} - 2B^{2k}} = sf|_{3B^{2k} - 2B^{2k}}$. Define $g = pr_2 \cdot s$. This map has properties (1)–(4).

The proof of the following lemma is similar to the proof of Lemma 3.

LEMMA 4. *Let $(0, 0)$, $(0, -1)$, $(0, 1)$ be the only fixed points for $f: 3B^{2k} \times I \rightarrow R^{2k} \times I$, and let $f(3B^{2k} \times I - 2B^{2k} \times I) \cap B^{2k} \times I = \emptyset$. Suppose that $f(3B^{2k}, t) \subset (R^{2k}, t)$, $t = \pm 1$, and suppose the index of each of these fixed points is 1. Then there exists a map $g: 3B^{2k} \times I \rightarrow R^{2k} \times I$ such that*

- (1) $g|_{(3B^{2k} - 2B^{2k}) \times I} = f|_{(3B^{2k} - 2B^{2k}) \times I}$.
- (2) $B^{2k} \times I$ *is invariant and complement invariant under g .*
- (3) $\{0\} \times I$ *is the fixed point set of g .*
- (4) $g|_{B^{2k} \times I}$ *is given by $g(x, t) = (-x, t)$.*

NOTATION. Let $x_0 \in \text{int } M$ be an isolated fixed point for the map $f: M \rightarrow M$. Let $U \subset V \subset M$ be a pair of euclidean neighborhoods of x_0 such that $f(U) \subset V$ and x_0 is the only fixed point for $f|_U$. We define the *index* of x_0 under f to be the index of $f|_U$.

REMARK. The index of the isolated fixed point $x_0 \in \text{int } M$ for $f: M \rightarrow M$ is well defined, i.e., the index does not depend on the choice of the pair of euclidean neighborhoods $U \subset V \subset M$.

THEOREM 1. *Suppose $\chi M \geq 0$. Then there exists a surjective self map $\phi: M \rightarrow M$ whose fixed point set has components F_i , with closed euclidean neighborhoods U_i given by homeomorphisms*

$$\begin{aligned} h_i: B^{2k} &\rightarrow U_i, & \text{if } m = 2k, \\ h_i: B^{2k} \times I &\rightarrow U_i, & \text{if } m = 2k + 1, \end{aligned}$$

$i = 1, \dots, \chi M$, such that

- (1) U_i *is invariant and complement invariant for all i .*
- (2) $U_i \cap U_j = \emptyset$ *unless $i = j$.*

- (3) $h_i^{-1} \cdot \phi \cdot h_i: B^{2k} \rightarrow B^{2k}$, is given by $x \mapsto -x$, if $m = 2k$, and $h_i^{-1} \cdot \phi \cdot h_i: B^{2k} \times I \rightarrow B^{2k} \times I$ is given by $(x, t) \mapsto (-x, t)$, if $m = 2k + 1$, for all i .
 (4) $h_i(x, t) \in \text{bd } M$ if $t = \pm 1$ and $m = 2k + 1$, for all x and i .

PROOF. Case (i). $\text{bd } M = \emptyset$. If M is open apply Lemma 2. If M is closed and $m = 2k + 1$ we conclude by Poincaré duality that $\chi M = 0$ and apply Lemma 2. If M is closed and $m = 2k$ we apply Lemma 2 to get a map $f_0: M \rightarrow M$ with at most one fixed point x_0 . R. Brown [2] shows that f_0 is ϵ -homotopic to a map f_1 with χM fixed points x_0^i , each of index 1, which are sufficiently far apart with respect to ϵ . This means that each fixed point x_0^i has a euclidean neighborhood U_i given by a homeomorphism $h_i: R^{2k} \rightarrow U_i$ such that (1) $h_i(0) = x_0^i$, (2) $U_i \cap U_j = \emptyset$, (3) $f_1 \cdot h_i(3B^{2k}) \subset U_i$, (4) $f_1(M - h_i(2B^{2k})) \cap h_i(B^{2k}) = \emptyset$. Observe that $h_i^{-1} \cdot f_1 \cdot h_i: 3B^{2k} \rightarrow R^{2k}$ satisfies the hypotheses of Lemma 3. As a result

$$\begin{aligned} \phi(x) &= f_1(x), \quad x \in M - \bigcup_{i=1}^{\chi M} U_i, \\ &= h_i \cdot g_i \cdot h_i^{-1}(x), \quad x \in U_i, i = 1, \dots, \chi M, \end{aligned}$$

satisfies properties (1)–(4) where $g_i: 3B^{2k} \rightarrow R^{2k}$ is given by Lemma 3.

Case (ii). $\text{bd } M \neq \emptyset$. Let $f_0: M \rightarrow M$ be the map given by Lemma 2. If $m = 2k$ then $\dim \text{bd } M = 2k - 1$ and f_0 is fixed point free on $\text{bd } M$ by Lemma 2. If M is open then we may apply Lemma 2. If M is compact let $Df_0: DM \rightarrow DM$ be the induced map on the double of M . Since $\chi DM = 2\chi M - \chi \text{bd } M = 2\chi M$ we may proceed as in Case (i). It remains to consider the case $\text{bd } M \neq \emptyset$, M compact, $m = 2k + 1$. Since $0 = \chi DM = 2\chi M - \chi \text{bd } M$ R. Brown [2] says that f_0 is ϵ -homotopic to a map f_1 with χM interior fixed points x_0^i , $i = 1, 2, \dots, \chi M$ and $2\chi M$ boundary fixed points x_{-1}^i, x_1^i , $i = 1, 2, \dots, \chi M$ each of index 1, which are sufficiently far apart with respect to ϵ . The map f_1 also has the property that $f_1(\text{bd } M) \subset \text{bd } M$. This means that each fixed point set $\{x_0^i, x_{-1}^i, x_1^i\}$, $i = 1, 2, \dots, \chi M$, has a euclidean neighborhood U_i given by a homeomorphism $h_i: R^{2k} \times I \rightarrow U_i$, $i = 1, 2, \dots, \chi M$, such that (1) $h_i(0, 0) = x_0^i$, $h_i(0, \epsilon) = x_\epsilon^i$, $\epsilon = \pm 1$, are the only fixed points of $f|U_i$, (2) $U_i \cap U_j = \emptyset$, unless $i = j$, (3) $h_i(R^{2k}, t) \subset \text{bd } M$, if $t = \pm 1$, (4) $f_1 \cdot h_i(3B^{2k} \times I) \subset U_i$, (5) $f_1(M - h_i(2B^{2k} \times I)) \cap h_i(B^{2k}) = \emptyset$. We may now apply Lemma 4 to complete the proof of the theorem.

COROLLARY 1. If $\chi M \geq 0$ then M does not have CFP²S.

PROOF. In [4] it is shown that there exist self maps f and g of $[-1, 1]$ which commute without common fixed points and which interchange -1 and 1 .

Define $f_1, g_1: B^{2k} \rightarrow B^{2k}$ by

$$f_1(x_1, x_2, \dots, x_{2k}) = (\alpha f(x_1/\alpha), -x_2, \dots, -x_{2k})$$

and

$$g_1(x_1, x_2, \dots, x_{2k}) = (\alpha g(x_1/\alpha), -x_2, \dots, -x_{2k}),$$

where $\alpha = (1 - \sum_{i=2}^{2k} x_i^2)^{1/2}$ and it is understood that $\alpha=0$ implies $\alpha f(x_1/\alpha) = 0 = \alpha g(x_1/\alpha)$. Observe that $f_1|_{\text{bd } B^{2k}}(x) = g_1|_{\text{bd } B^{2k}}(x) = -x$. Define $\tilde{f}_1, \tilde{g}_1: B^{2k} \times I \rightarrow B^{2k} \times I$ by $\tilde{f}_1(x, t) = (f_1(x), t)$, $\tilde{g}_1(x, t) = (g_1(x), t)$. Let ϕ, h_i , and U_i be the maps and sets given by Theorem 1. Define F and G , self maps of M , by:

$$\begin{aligned} F(x) &= \phi(x), & \text{if } x \in M - \bigcup_{i=1}^{x^M} U_i, \\ &= h_i \cdot f_1 \cdot h_i^{-1}(x), & \text{if } x \in U_i, \quad m = 2k, \\ &= h_i \cdot \tilde{f}_1 \cdot h_i^{-1}(x), & \text{if } x \in U_i, \quad m = 2k + 1, \end{aligned}$$

and

$$\begin{aligned} G(x) &= \phi(x), & \text{if } x \in M - \bigcup_{i=1}^{x^M} U_i, \\ &= h_i \cdot g_1 \cdot h_i^{-1}(x), & \text{if } x \in U_i, \quad m = 2k, \\ &= h_i \cdot \tilde{g}_1 \cdot h_i^{-1}(x), & \text{if } x \in U_i, \quad m = 2k + 1. \end{aligned}$$

Maps F and G have the desired properties.

COROLLARY 2. *Projective spaces over the real, complex, and quaternion fields do not have CFP²S.*

PROOF. Projective spaces of odd dimension over the real and complex fields do not have the fixed point property so do not have CFP²S. The other projective spaces do have this property (see [3, pp. 18, 22]), but also have positive Euler characteristic, so apply Corollary 1.

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