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SUMMING CLOSED U-SETS FOR WALSH SERIES

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ABSTRACT. The union of countably many closed sets of uniqueness for the Walsh series is again a set of uniqueness.

In a classic paper on Walsh series [1] Šneĭder proved the finite union of closed sets of uniqueness, E_i, for the Walsh series is again a set of uniqueness. In case there were countably many sets of uniqueness he needed to further assume that $E_i \subset V_i$, where $V_1, \cdots, V_r, \cdots$, formed a disjoint collection of open intervals. Combining recent developments in Walsh series [2] with a slight generalization of a classical lemma (see Lemma 3) we will show this further assumption is unnecessary. The proof is similar to the proof of the trigonometric analogue given in [3].

We briefly review the definition of the Walsh functions: let $\phi_0(x) = 1$ if $0 \leqq x < \frac{1}{2}$ and $\phi_0(x) = -1$ if $\frac{1}{2} \leqq x \leqq 1$. Extend ϕ_0 by periodicity (of period 1) to the whole real line and define $\phi_n(x) = \phi_0(2^n x)$. Finally, the first Walsh function is $\Psi_0(x) \equiv 1$, and given any $n > 0$, then the $(n+1)$th Walsh function is $\Psi_n(x) = \phi_{n_1}(x) \cdots \phi_{n_r}(x)$ where $n = \sum_{i=1}^{r} 2^{n_i}$ uniquely determines the n_i by specifying $n_{i+1} < n_i$.

A set $E \subseteq [0, 1]$ will be called a set of uniqueness for the Walsh series (U-set for brevity) if the only Walsh series $S(x) = \sum_{k=0}^{\infty} a_k \Psi_k(x)$, satisfying

$$\lim_{n\to\infty} S_n(x) \equiv \lim_{n\to\infty} \sum_{k=0}^{n-1} a_k \Psi_k(x) = 0 \qquad \text{for } x \notin E$$

is the zero series.

The following lemma appeared as Theorem 4 in [1].

LEMMA 1. *If S is a Walsh series whose coefficients, a_k, tend to zero as k tends to ∞ and I is an interval with dyadic rational endpoints then there is a Walsh series S^* which is equiconvergent with S on I and uniformly convergent to zero on $[0, 1] \sim I$.*

The following lemma is a corollary of Theorem 2 in [2].

LEMMA 2. *If $S(x) = \sum a_k \Psi_k(x)$ is a Walsh series such that*

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(i) $\lim_{n\to\infty} S_n(x)=0$ *a.e.* $x\in[0, 1]$;

(ii) $\lim\sup_{n\to\infty} |S_n(x)|<\infty$ *except possibly on a countable subset of* $[0, 1]$;

then $a_k=0$ *for* $k=0, 1, 2, \cdots$.

COROLLARY. *Let E be a closed U-set contained in an open interval J. If* $S(x)=\sum_{k=0}^{\infty} a_k\Psi_k(x)$ *is a Walsh series such that*

(i) $\lim\sup_{n\to\infty} |S_n(x)|<\infty$ *for* $x\in J\sim E$;

(ii) $\lim_{n\to\infty} S_n(x)=0$ *a.e.* $x\in J$;

then S converges to zero for every $x\in J$.

PROOF. We first notice that hypothesis (ii) implies $a_k\Psi_k(x)\to 0$ a.e. $x\in J$. But $|\Psi_k(x)|=1$ for any dyadic irrational x so we have $a_k\to 0$ as $k\to\infty$. (This holds for any series satisfying (ii).)

Now E is closed, so let J_1 be any subinterval of J which is disjoint with E. Let I be a subinterval of J_1 with dyadic rational endpoints. Use Lemma 1 to construct the series S^*.

Then since $I\cap E=\emptyset$, we apply Lemma 2 to S^* to conclude $S^*\equiv 0$.

But S and S^* are equiconvergent, so S converges to zero in I. Since I and J_1 were arbitrary we conclude S converges to zero in $J\sim E$.

Finally, if I is any subinterval of J with dyadic rational endpoints then applying Lemma 1 again we conclude S^* converges to zero on $[0, 1]\sim E$. But E is a U-set so $S^*\equiv 0$ and thus S converges to zero everywhere on I. But I was arbitrary, so S converges to zero on J.

LEMMA 3. *Let f_n $(n=0, 1, \cdots)$ be a function which is continuous in* $[0, 1]$ *except on a finite set Z_n. Then the set*

$$N = \left\{x:\limsup_{n\to\infty} |f_n(x)| = \infty\right\}$$

is empty, countable, or of the second category on itself.

PROOF.

$$N = \bigcap_k \{x: |f_n(x)| > k \text{ for some } n\}$$
$$\equiv \bigcap_k N_k.$$

By hypothesis each N_k is the union of an open set V_k with a subset $Z_k^*\subseteq Z_k$. If $\bigcap V_k$ is empty or countable then $N\subseteq\bigcap V_k\cup\bigcap Z_k$ is empty or countable. If $\bigcap V_k$ is uncountable then it is of the second category on itself [3, p. 349] and since it is a subset of N, N is of the second category on itself.

THEOREM. *The countable union of closed U-sets is a U-set.*

PROOF. Let $E_1, E_2, \cdots$ be closed U-sets and suppose S is a Walsh series convergent to zero outside $E \equiv \bigcup_i E_i$. Šneĭder [1] proved all U-sets have measure zero, and thus E also has measure zero. In particular, S converges to zero a.e.

Let $N = \{x : \limsup_{n\to\infty} |S_n(x)| = \infty\}$, and suppose N is of the second category on itself.

Then by defining $N_i = N \cap E_i$ we conclude there is an open interval J and index i_0 such that $N_{i_0} \cap J$ is dense in $N \cap J \neq \emptyset$. But E_{i_0} is closed and $N_{i_0} = N \cap E_{i_0}$ so $N \cap J = E_{i_0} \cap N \cap J \subseteq E_{i_0} \cap J$.

We may assume the endpoints of J are not in E_{i_0}, and thus that $E_{i_0} \cap J$ is a closed U-set contained in J. Furthermore, if $x \notin E_{i_0} \cap J$ then $x \notin N \cap J$ so the partial sums of S are bounded in J outside $E_{i_0} \cap J$. Thus by the corollary S converges to zero everywhere in J. Thus $J \cap N = \emptyset$ which contradicts the choice of J.

Thus N cannot be of the second category on itself. Lemma 3 forces N to be countable or empty which by Lemma 2 implies $S \equiv 0$ as was to be shown.

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