PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 29, Number 1, June 1971

GEOMETRIC THEORY OF DIFFERENTIAL EQUATIONS. II: ANALYTIC INTERPRETATION OF A GEOMETRIC THEOREM OF BLASCHKE

H. GUGGENHEIMER[1]

ABSTRACT. According to Blaschke, a C^2 closed convex plane curve admits at least three pairs of points with parallel tangents and equal curvature radii. We generalize the result to theorems on Hill equations with either first or second intervals of stability collapsed. We also prove a four-extrema theorem for second order linear differential equations without periodicity properties.

1. We consider a second order differential equation

$$x'' + (q(t) + \lambda)x = 0, \qquad -\infty < t < \infty, \tag{1}$$

that is oscillatory for $\lambda > \lambda_0$. Following Borůvka, the dispersion $\phi(t_0, \lambda)$ is defined as the value t_1 of the first zero following t_0 of any solution $x(t)$ that vanishes at t_0 without being identically zero ($x(t_0) = 0$, $x'(t_0) \neq 0$). In other words, λ is the first eigenvalue of (1) for the problem $x(t_0) = x(\phi(t_0, \lambda)) = 0$. Higher dispersions are defined by $\phi = \phi_1$, $\phi_k(t, \lambda) = \phi(\phi_{k-1}(t, \lambda))$.

A special case of interest are the Hill equations, for which $q(t+T) = q(t)$, T constant. The kth interval of instability of a Hill equation is the set of λ for which there exists at least one τ so that $\phi_k(\tau, \lambda) = \tau + T$, i.e., for which λ is the kth eigenvalue for the problem

$$x'' + (q(t) + \lambda)x = 0, \quad x(\tau) = x(\tau + T) = 0, \qquad \tau \leqq t \leqq \tau + T.$$

We put $\lambda_k = \inf \lambda$, $\Lambda_k = \sup \lambda$. It is well known that $\Lambda_{k-1} < \lambda_k \leqq \Lambda_k < \lambda_{k+1}$. The kth interval of instability collapses ($\lambda_k = \Lambda_k$) if there exists a λ^* so that

$$\phi_k(t, \lambda^*) = t + T, \qquad -\infty < t < \infty. \tag{2}$$

It follows from the theorem on p. 123 of [2] that if (2) holds for an equation (1) we also have $q(\phi_k(t)) = q(t)$ and, hence, *a relation* (2) *is possible only for a Hill equation.*

2. W. Blaschke has indicated [1] the following theorem: Any C^2 closed, convex curve admits three distinct pairs of points with parallel

Presented to the Society, June 8, 1970; received by the editors May 28, 1970.
AMS 1970 *subject classifications.* Primary 34A30; Secondary 52A10.
Key words and phrases. Hill equation, dispersion, periodic function.
[1] Research partially supported by NSF Grant GP-19133.

tangents and equal curvature radii. We denote by θ the angle between the positive x-axis and the positively oriented tangent to the curve. A positive function $\rho(\theta)$ is the radius of curvature of a closed, convex curve iff

$$\rho(0) = \rho(2\pi),$$
$$\int_0^{2\pi} \rho(\theta) \cos \theta \, d\theta = \int_0^{2\pi} \rho(\theta) \sin \theta \, d\theta = 0.$$

Blaschke's theorem then asserts the existence of three distinct values $\theta_1, \theta_2, \theta_3$ between 0 and π so that $\rho(\theta_i) = \rho(\theta_i + \pi)$, $i = 1, 2, 3$. Note that $x(\theta) = (\cos \theta, \sin \theta)$ is a solution vector of the Hill equation $x'' + x = 0$. For that equation, $\phi(t) = t + \pi$ and $\int_0^{\phi_2(0)} \rho(t)x(t)dt = 0$ implies $\rho(t_i) = \rho(\phi(t_i))$ for $i = 1, 2, 3$. In this form, Blaschke's theorem has many geometrical applications [3]. We show that the theorem can be generalized to other differential equations. We use the notation $f(h(t)) = f \circ h(t)$.

3. In the following we assume that the coefficient $q(t)$ in (1) is either Lebesgue integrable on any interval of length T or is the generalized derivative of a one-sided continuous function.

THEOREM. *Given a Hill equation with collapsed second interval of stability* ($\phi_2(t, \lambda) = t + T$) *and a function* $f(t)$ *continuous on some interval* $\tau \leqq t \leqq \tau + T$. *If* $f(\tau) = f(\tau + T)$ *and* $\int_\tau^{\tau+T} f(t)x(t)dt = 0$ *for all solutions of the Hill equation then there exist at least three distinct values* t_1, t_2, t_3 *in* $\tau \leqq t < \phi(\tau)$ *for which*

$$f(t_i)\phi'(t_i)^{-3/4} = f \circ \phi(t_i)\phi' \circ \phi(t_i)^{-3/4}, \qquad i = 1, 2, 3.$$

The integral condition holds if it is true for two linearly independent solutions of (1). For an arbitrary solution vector $x = (x_1, x_2)$ of (1) we have

$$\phi'(t) = |x(\phi(t))|^2 / |x(t)|^2 > 0$$

and, for $x_i(t) \neq 0$ $(i = 1, 2)$, also

$$\phi'(t) = x_i \circ \phi(t)^2 / x_i(t)^2 \quad \text{or} \quad x_i \circ \phi(t) = - x_i(t)\phi'(t)^{1/2}.$$

This is proved for continuous $q(t)$ in [2, p. 113], and in the general case in [5, 3.1]. In any case ϕ' is differentiable and the hypothesis on ϕ_2 implies $\phi' \circ \phi(t) = \phi'(t)^{-1}$. We put $g(t) = f(t)\phi'(t)^{-3/4}$, and $D(t) = (g(t) - g \circ \phi(t))\phi'(t)^{3/4}$. Then

$$\begin{aligned}\int_\tau^{\phi(\tau)} D(t)x(t)dt &= \int_\tau^{\phi(\tau)} f(t)x(t)dt - \int_\tau^{\phi(\tau)} f\circ\phi(t)(\phi'\circ\phi(t))^{-3/4}\phi'(t)^{3/4}x(t)dt \\ &= \int_\tau^{\phi(\tau)} f(t)x(t)dt - \int_\tau^{\phi(\tau)} f\circ\phi(t)\phi'(t)^{1/2}x(t)\phi'(t)dt \\ &= \int_\tau^{\phi_2(\tau)} f(t)x(t)dt = 0.\end{aligned}$$

We may assume that $D(t)$ does not vanish identically, otherwise there is nothing to prove. We take x_1 as the fundamental solution $x_1(\tau)=0$, $x_1'(\tau)=1$. The function $x_1(t)$ is positive in $\tau<t<\phi(\tau)$ and

$$\int_\tau^{\phi(\tau)} D(t)x_1(t)dt = 0$$

implies that the continuous function $D(t)$ must change sign at least once in $\tau<t<\phi(\tau)$. We denote the abscissa at which D changes sign by τ^*. We choose x_2 as a solution of (1) that vanishes at τ^*. By the monotonicity property of ϕ, τ^* is the only zero of $x_2(t)$ in $\tau<t<\phi(\tau)$. From

$$\int_\tau^{\phi(\tau)} D(t)x_2(t)\,dt = \int_\tau^{\tau^*} Dx_2\,dt + \int_{\tau^*}^{\phi(\tau)} Dx_2\,dt = 0$$

we see that D must change sign at least another time in $\tau<t<\phi(\tau)$. But sign $D(\phi(\tau)) = -$sign $D(\tau)$ and either D changes sign an odd number of times or $D(\tau)=0$. In both cases the theorem holds. The proof follows Süss's proof of Blaschke's problem [6].

If the given equation has collapsed first interval of stability, the same equation for the period $2T$ has collapsed second interval of stability and, in addition, $\phi'(t)=1$. We obtain the

COROLLARY. *Given a Hill equation with collapsed first interval of stability* ($\phi(t)=t+T$) *and a function* $f(t)$ *continuous on some interval* $\tau\leqq t\leqq\tau+2T$. *If*

$$f(\tau) = f(\tau+2T) \quad \textit{and} \quad \int_\tau^{\tau+2T} f(t)x(t)dt = 0$$

for all solutions of the Hill equation then there exist at least three distinct values t_1, t_2, t_3 *in* $\tau\leqq t<\tau+T$ *for which*

$$f(t_i) = f(t_i+T), \qquad i = 1, 2, 3.$$

This is a direct generalization of Blaschke's theorem.

4. The second dispersion is useful in other geometric theorems. The graph $x(t)=(x_1(t), x_2(t))$ whose coordinate functions are solutions of the linear differential equation

$$x'' + a_1(t)x' + a_2(t)x = F(t), \tag{3}$$

is starshaped from the origin for any interval $\tau \leqq t < \phi_2(\tau)$. The following theorem is a known generalization of the Four Vertices Theorem [4, Proposition 4.1]: Let $f(t)$, $g(t)$ be continuous functions on $\tau \leqq t \leqq \tau^*$, $g(t)>0$ and $f(\tau)=f(\tau^*)$, $g(\tau)=g(\tau^*)$. If $v(t)$ is a continuous, star-shaped, plane arc and

$$\int_\tau^{\tau^*} f(t)v(t)dt = \int_\tau^{\tau^*} g(t)v(t)dt = 0,$$

then $f(t)/g(t)$ has at least four relative extrema in $\tau \leqq t < \tau^*$. We obtain the result:

Let $\phi_2(t)$ be the second dispersion of a second order linear differential equation (3) *with integrable coefficients. Let $f(t)$, $g(t)$ be continuous functions, periodic of period $T \leqq \phi_2(\tau)$ and assume $g(t)>0$ and*

$$\int_\tau^{\tau+T} f(t)x(t)dt = \int_\tau^{\tau+T} g(t)x(t)dt = 0$$

for any absolutely continuous solution $x(t)$ of (3). *Then f/g has at least four relative extrema in $\tau \leqq t < \tau + T$.*

Bibliography

1. W. Blaschke, *Aufgabe* 540, Arch. Math. Phys. (3) **26** (1917), 65.

2. O. Borůvka, *Lineare Differentialtransformationen* 2. *Ordnung*, Hochschulbücher für Mathematik, Band 67, VEB Deutscher Verlag der Wissenschaften, Berlin, 1967. MR **38** #4743.

3. H. Guggenheimer, "Geometrical applications of integral calculus," in K. O. May, *Lectures on calculus*, Holden-Day, San Francisco, Calif., 1967, pp. 75–93.

4. ———, *Sign changes, extrema, and curves of minimal order*, J. Differential Geometry **3** (1969), 511–521.

5. ———, *Geometric theory of differential equations*. I. *Second order linear equations*, SIAM J. Math. Anal. (to appear).

6. W. Süss, *Lösung der Aufgabe* 540, Arch. Math. Phys. **26**, Jber. Deutsch. Math.-Verein. **33** (1924), 32–33.

Polytechnic Institute of Brooklyn, Brooklyn, New York 11201