PROPER MAPPINGS AND DIMENSION

JAMES KEESLING

ABSTRACT. In this note it is shown that if W is the long line and f is a proper mapping of $W \times [0, 1]$ into Y , then $\dim Y \geq 2$. This answers a question raised by Isbell.

The main result of this note is that if W is the long line and f is a proper mapping of $W \times [0, 1]$ into Y , then $\dim Y \geq 2$. This answers a question posed by Isbell [2, pp. 119–120]. The main result will follow after a few preliminary results.

LEMMA. *Let $X = [0, 1] \times [0, 1]$ with $C_0 = \{0\} \times [0, 1]$ and $C_1 = \{1\} \times [0, 1]$. If D is a closed set which separates C_0 and C_1 in X , then D has a noncompact component.*

PROOF. Suppose that $X - D = U \cup V$ is a separation with $C_0 \subset U$ and $C_1 \subset V$. There is a continuous $f: U \cup D \rightarrow [0, 1]$ which is 0 on C_0 and 1 on D . There is a continuous function $g: V \cup D \rightarrow [1, 2]$ which is 1 on D and 2 on C_1 . Thus there is a continuous function $h: X \rightarrow [0, 2]$ which is 0 on C_0 , 1 on D , and 2 on C_1 . Accordingly there is a homeomorphism $i: X \rightarrow X$ taking C_0 to C_0 , C_1 to C_1 , and D to a set E whose closure in R^2 is disjoint from $\overline{C_0}$ and $\overline{C_1}$. Then $\overline{E}$ separates the two sides of $[0, 1] \times [0, 1]$ and thus one of its components K intersects both the top and bottom of the square. Now K contains a component of E and it cannot contain a compact component of E . Therefore E and thus D have noncompact components.

PROPOSITION. *Let X , C_0 , and C_1 be as in the Lemma. If $f(X) = Y$ is a proper mapping with $f(C_0) \cap f(C_1) = \emptyset$, then $\dim Y \geq 2$.*

PROOF. We note that Y is separable metric by [6]. Thus the various dimension functions coincide as proved in [1]. We will show that $\text{Ind } Y \geq 2$ by showing that $f(C_0)$ and $f(C_1)$ cannot be separated in Y by a closed 0-dimensional set. Suppose that E is closed and separates $f(C_0)$ and $f(C_1)$ in Y . Then $f^{-1}(E)$ is closed in X and separates C_0 and C_1 . By the Lemma, $f^{-1}(E)$ has a noncompact component D . Since f is proper and D is closed, f cannot be constant on D . Therefore $f(D)$ is connected and not a point and thus $\dim f(D) \geq 1$ and $\dim E \geq 1$. This proves the Proposition.

Received by the editors July 22, 1970.

AMS 1970 subject classifications. Primary 54F45; Secondary 54C10.

Key words and phrases. Dimension, proper mapping, locally compact space.

Copyright © 1971, American Mathematical Society

THEOREM. *If $f: W \times [0, 1) \rightarrow Y$ is proper, then Y contains a compact metric subspace D with $\dim D \geq 2$.*

PROOF. Let $A_0 = \{0\} \times [0, 1 - 2^{-1}]$. Using the fact that $f^{-1}f(A_0)$ is compact, let $A_1 = [0, \alpha_1] \times [0, 1 - 2^{-n_1}]$ be such that $f^{-1}f(A_0) \subset \text{int } A_1$. Continue inductively defining $A_i = [0, \alpha_i] \times [0, 1 - 2^{-n_i}]$ such that $f^{-1}f(A_i) \subset \text{int } A_{i+1}$ for all i . Let $B = \bigcup_{i=1}^{\infty} A_i = \bigcup_{i=1}^{\infty} f^{-1}f(A_i)$. Then B is an inverse set and $B = [0, \gamma) \times [0, 1)$ where $\gamma = \sup \{\alpha_i\} = \lim \alpha_i$. Now $\overline{B} = [0, \gamma] \times [0, 1)$ and $f|_{\overline{B}}$ is proper. Letting $C_0 = \{0\} \times [0, 1)$ and $C_1 = \{\gamma\} \times [0, 1)$ we have that $f(C_0) \cap f(C_1) = \emptyset$. Applying the Proposition we get that $\dim f(\overline{B}) \geq 2$. Since $f(\overline{B})$ is separable metric and locally compact it contains a compact metric D with $\dim D \geq 2$.

COROLLARY. *There is a space Y which has the property that $\min \dim(Y - \{p\}) = \min \dim Y - 2$ for some point $p \in Y$.*

An outline of the proof of the Corollary assuming the Theorem is given on p. 120 of [2]. There is no need to elaborate here.

REMARK. The cited proof outline of Isbell to the Corollary requires a space X which is locally compact and has the property proved for $W \times [0, 1)$ in the Theorem of this paper. Such a space cannot be paracompact (and consequently not metric) since such a space would be the free union of locally compact Lindelöf spaces S_α . Each S_α admits a proper mapping into $[0, \infty)$ and thus the free union of the S_α 's admits a proper mapping into the free union of an appropriate collection of half-lines.

If local compactness were not needed, then there are separable metric spaces satisfying this condition. Take for example any separable metric space X which is totally disconnected and n -dimensional. Such spaces exist for every $n \geq 1$ [3, p. 152]. In [4] such spaces are constructed which are also complete. For such a space X , if $f(X) = Y$ is proper, then Y is separable metric by [6]. Also for each $y \in Y$, $\dim f^{-1}(y) = 0$, since $f^{-1}(y)$ is compact and totally disconnected [1, p. 22]. By the Hurewicz theorem for closed mappings which lower dimension [1, p. 91], we must have $\dim Y \geq \dim X = n$. That is, the dimension of such a space cannot be lowered by a proper mapping.

REFERENCES

1. W. Hurewicz and H. Wallman, *Dimension theory*, Princeton Math. Series, vol. 4, Princeton Univ. Press, Princeton, N.J., 1941. MR 3, 312.
2. J. Isbell, *Uniform spaces*, Math. Surveys, no. 12, Amer. Math. Soc., Providence, R.I., 1964. MR 30 #561.
3. K. Kuratowski, *Topologie*. Vol. 2, Monografie Math., Tom 21, PWN, Warsaw,

1961; English transl., Academic Press, New York; PWN, Warsaw, 1968. MR 24 #A2958.

4. A. Lelek, *Some problems in metric topology*, Mimeographed Lecture Notes, Louisiana State University, Baton Rouge, La., 1965/66.

5. J. Nagata, *Modern dimension theory*, Bibliotheca Math., vol. 6, Interscience, New York, 1965. MR 34 #8380.

6. A. H. Stone, *Metrizability of decomposition spaces*, Proc. Amer. Math. Soc. 7 (1956), 690-700. MR 19, 299.

UNIVERSITY OF FLORIDA, GAINESVILLE, FLORIDA 32601