PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 29, Number 1, June 1971

STABLE THICKENINGS IN THE TOPOLOGICAL CATEGORY

R. L. CHAZIN[1]

ABSTRACT. A *thickening*, in the topological category, of a complex K is an equivalence class of simple homotopy equivalences $\phi: K \to M$, where M is a topological manifold with boundary. Here it is shown that for stable thickenings (dim $M \gg$ dim K), the set $\mathfrak{I}(K)$ of stable thickenings is in 1-1 correspondence with homotopy classes of maps of K into BTop.

Wall [1] and Mazur [2] have studied a "functor" of complexes called a *thickening*. Given a complex K, a thickening of K is essentially an m-manifold M which is homotopy equivalent to K. The set of these, under a suitable equivalence relation, forms a set $\mathfrak{I}^m(K)$. It is clear that this kind of construction can be done in the differentiable, piecewise-linear, or topological categories.

In [1] and [2] it is shown that the stable thickenings of a complex K are a representable functor, i.e. if we denote the stable thickenings of K by $\mathfrak{I}(K)$, then we have $\mathfrak{I}(K) \approx [K, \mathrm{BO}]$ in the smooth category and $\mathfrak{I}(K) \approx [K, \mathrm{BPL}]$ in the piecewise-linear category. In this note we establish the analogous result for the topological category. In a subsequent paper, we will give an analogous result for the homotopy category.

1. **Definition.** We are able to use the same definition as Wall. Let K be a finite complex of dimension k with basepoint $*$, and $\phi: K \to M$, a simple homotopy equivalence of K into a compact topological manifold-with-boundary of dimension m, $m \geqq k+3$. The notion of *simple* homotopy equivalence is well defined in the topological category since, by Kirby-Siebenmann [3], every compact topological manifold has the homotopy type of a finite complex.

We require that the basepoint $*$ of M lie in ∂M and that the inclusion $i: \partial M \subset M$ induce an isomorphism $i_*: \pi_1(\partial M) \to \pi_1(M)$, and that the tangent space of M at $*$ be oriented. Then we say that the pair (M, ϕ) defines a *pre-m-thickening* of K.

Define two pre-thickenings (M_1, ϕ_1), (M_2, ϕ_2) of K to be equivalent, if there is a (topological) homeomorphism $h: M_1 \to M_2$, preserving $*$ and the given orientations of the tangent space there, such that

Received by the editors October 22, 1969.

AMS 1970 *subject classifications*. Primary 57A15; Secondary 55D15, 57C10.

Key words and phrases. Thickening, stable thickening, topological manifold, classification of stable thickenings.

[1] Partially supported by NSF Grant GP-7913.

$h\phi_1 \sim \phi_2 : (K, *) \to (M_2, *)$. An *m-thickening* of K will be such an equivalence class and by $\mathfrak{I}^m(K)$ we denote the set of all such m-thickenings.

2. **Stable thickenings.** As in Wall [1, §5] denote by $\mathfrak{I}(K)$ the stable limit of the inclusions $\mathfrak{I}^m(K) \to \mathfrak{I}^{m+1}(K)$. $\mathfrak{I}(K)$ will be called the set of *stable thickenings* of K.

Given a representative manifold M in $\mathfrak{I}(K)$ we take its tangent microbundle and pass to the classifying space. Thus we get a classifying map $M \to \text{BTop}$ and hence a map $K \to \text{BTop}$, which is basepoint preserving and unique up to homotopy. Thus we have a natural map $\tau(K) : \mathfrak{I}(K) \to [\text{K}, \text{BTop}]$. We can now state the following theorem.

THEOREM 1. *For any K, $\tau(K)$ is a bijection.*

The remainder of this paper is devoted to the proof of this theorem.

As in the smooth and PL cases, $\tau(*)$ is the homotopy class of the constant map $K \to \text{BTop}$. Let $\phi : K \to M_0$ be the trivial thickening, i.e. the one corresponding to the map $K \to \text{pt.}$; M_0 is parallelizable and hence corresponds to the constant map $K \to \text{BTop}$.

The proof that τ is surjective in the topological category is exactly the same as in the smooth and PL categories; see Wall [1, p. 80]. For completeness, we restate it here.

Let $f : K \to \text{BTop}$. As Top is the limit of the Top_n, f can be factored as

$$K \xrightarrow{\hat{f}} \text{BTop}_n \xrightarrow{j} \text{BTop}$$

where j is inclusion, for some n. Thus f induces a bundle over K with fibre R^n. Since ϕ is a homotopy equivalence there is therefore a corresponding bundle ξ over M_0, a trivial thickening for K. Then $\tau(E(\xi)) = \pi^*(\xi) \oplus \epsilon^m$, where π is the projection of ξ and ϵ^m a trivial bundle. Let $\mathfrak{X}$ be the zero section in ξ. Then the thickening determined by ϕ followed by a map into a compact neighborhood of $\mathfrak{X}$ is a thickening α such that $\tau(\alpha) = [f]$. The latter assertion follows immediately from the equation

$$\tau(E(\xi)) = \pi^*(\xi) \oplus \epsilon^m,$$

and the fact that $\mathfrak{X}$ is a homotopy equivalence.

To prove that τ is injective we can assume, after stabilizing, that there are thickenings (M_1^m, ϕ_1) and (M_2^m, ϕ_2) with equivalent tangent bundles. We then need the following lemma:

LEMMA 1. *Let $\phi : M_1^m \to M_2^m$ be such that $\phi^*\tau(M_2) \simeq (M_1)$. Then there is an immersion $\psi : M_1^m \to M_2^m$ such that $\psi \sim \phi$.*

PROOF. We use the immersion theorem of Lees [4] or Gauld [5]. For convenience in applying their results we give the proof in the language of microbundles.

Given ϕ, we construct a representation $\hat{\phi}:\tau(M_1)\to\tau(M_2)$. This is essentially a map such that the following diagram commutes:

$$\begin{array}{ccc} M_1 & \xrightarrow{\ \phi\ } & M_2 \\ \downarrow{\scriptstyle \Delta} & & \downarrow{\scriptstyle \Delta} \\ M_1 \times M_1 & \xrightarrow{\ \hat{\phi}\ } & M_2 \times M_2 \\ \downarrow{\scriptstyle \mathrm{pr}_1} & & \downarrow{\scriptstyle \mathrm{pr}_1} \\ M_1 & \xrightarrow{\ \phi\ } & M_2 \end{array}$$

and for which there exist charts

$$\beta: U \times R^m \to M_1 \times M, \qquad \gamma: U \times R^m \to M_1 \times M_2,$$

such that $\gamma^{-1}\hat{\phi}\beta = 1_U \times 1_{R^m}$.

We now apply Gauld's theorem. This essentially asserts that under certain very mild restrictions, which are satisfied here, homotopy classes of immersions of M_1 into M_2 are in 1-1 correspondence with homotopy classes of representations of $\tau(M_1)$ in $\tau(M_2)$. Hence ϕ is homotopic to some immersion ψ.

Since $m \geqq 2k+1$, ϕ_1 is homotopic to an embedding ϕ_1', by Dancis [6]. We can also modify $\psi\phi_1'$ by a regular homotopy to make it an embedding.

We now wish to compress M_1 into a neighborhood U of $\phi_1'(K)$. As there is no satisfactory theory of regular neighborhoods in the topological category, we use Lees' modification [7] of an engulfing theorem due to Newman [8]:

THEOREM 3 (LEES). *Let Q^q be an open topological manifold with $q \geqq 5$. Let U be an open subset of Q with (Q, U) $(q-3)$-connected. Suppose that any compact subset of Q lies inside a compact subset C' with $(Q, Q-C')$ 2-connected. Then any compact subset C of Q can be engulfed by U, i.e. there is a homeomorphism $h: Q\to Q$, fixed outside a compact set C'' with $h(U) \supset C$.*

Now attach a collar to M_1; let $M_1' = M_1 \cup \partial M \times I$. Then M_1 is locally flat in M_1'. A neighborhood U as required by Theorem 3 can be found by removing a suitable small closed subset and using duality or a cellular approximation theorem [9, 7.6.17]. Applying Theorem 3, we compress M_1 into U, i.e., 1_{M_1} is isotopic to an embedding of a neighborhood of M_1 in M_1', into U.

The remainder of the proof now proceeds as in Wall [1]: Since $\psi\phi_1'$ is an embedding of K, we can assume that ψ embeds $\phi_1'K$ and thus a neighborhood of $\phi_1'K$. Using the above compression, ψ is isotopic to a map ψ' which embeds M_1 in M_2. Applying the s-cobordism theorem, which holds in the topological category (using Kirby and Siebenmann [3]), we conclude that M_1 and M_2 are equivalent thickenings. This completes the proof.

References

1. C. T. C. Wall, *Classification problems in differential topology*. IV, Topology **5** (1966), 73–94. MR **33** #734.

2. B. Mazur, *Differential topology from the point of view of simple homotopy theory*, Inst. Hautes Études Sci. Publ. Math. No. 15 (1963). MR **28** #4550.

3. R. C. Kirby and L. C. Siebenmann, *On the triangulation of manifolds and the hauptvermutung*, Bull. Amer. Math. Soc. **75** (1969), 742–749. MR **39** #3500.

4. J. A. Lees, *Immersions and surgeries of topological manifolds*, Bull. Amer. Math. Soc. **75** (1969), 529–534. MR **39** #959.

5. D. Gauld, *Mersions of topological manifolds*, Thesis, University of California, Los Angeles, Calif., 1969.

6. J. Dancis, *Approximations and isotopies in the trivial range*, Topology Seminar (Wisconsin, 1965), Ann. of Math. Studies, no. 60, Princeton Univ. Press, Princeton, N. J., 1966. MR **36** #7144.

7. J. A. Lees, Thesis, Rice University, Houston, Tex., 1968.

8. M. H. A. Newman, *The engulfing theorem for topological manifolds*, Ann. of Math. (2) **84** (1966), 555–571. MR **34** #3557.

9. E. H. Spanier, *Algebraic topology*, McGraw-Hill, New York, 1966. MR **35** #1007.

UNIVERSITY OF CALIFORNIA, IRVINE, CALIFORNIA 92664