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UNIQUENESS OF TOPOLOGY FOR COMMUTATIVE SEMISIMPLE F-ALGEBRAS

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ABSTRACT. Let B be an F-algebra and A be a commutative semisimple F-algebra such that the spectrum of A contains no isolated points. We prove that any homomorphism of B onto A is necessarily continuous. Let A be a commutative semisimple algebra. We prove that there is at most one topology with respect to which A is an F-algebra.

Introduction. Let A be a semisimple algebra over the complex numbers. It was shown by Gelfand [3, Satz 17] that if A is commutative and has an identity, then there is at most one norm (up to equivalence) which makes A into a Banach algebra. C. E. Rickart gave several extensions of this result in the paper [6]. In [4] B. E. Johnson proved that the norm is unique in any semisimple Banach algebra.

E. A. Michael [5, §14] extended the theorem on uniqueness of topology to certain types of F-algebras. In particular it is proved in [5] that the topology is unique for commutative semisimple F-algebras in which every homomorphism of the algebra onto the complex numbers is continuous. (Whether all homomorphisms of a commutative F-algebra onto the complex numbers are continuous is one of the outstanding questions which remain open for F-algebras.) In this paper we show that if A is a commutative semisimple algebra, then there is at most one topology with respect to which A is an F-algebra. We also show that if ϕ is a homomorphism of an F-algebra B onto a commutative semisimple F-algebra A whose spectrum contains no isolated points, then ϕ is necessarily continuous.

Uniqueness of topology. A commutative F-algebra is a commutative algebra over the complex numbers which is a complete T_2 topological space with respect to a topology determined by a countable family of multiplicative seminorms $\{\|\cdot\|_i\}$, $i=1, 2, \cdots$. No generality is lost if the seminorms are assumed to be increasing. That is, we may assume $\|\cdot\|_i \leqq \|\cdot\|_{i+1}$ for $i=1, 2, \cdots$. The spectrum $M(A)$ of a commutative F-algebra A is the space of all continuous

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homomorphisms of A onto the complex numbers. $M(A)$ is given the Gelfand topology. If A is a commutative F-algebra, then A can be realized as the inverse limit of a sequence of commutative Banach algebras A_n [5]. The spectrum $M(A)$ of A is the union of the spectra $M(A_n)$ and each $M(A_n)$ is embedded continuously in $M(A)$ [1], [5]. A commutative F-algebra A is said to be semisimple if its radical is $\{0\}$. It is shown in [5] that A is semisimple if and only if for an element x in A, $\phi(x)=0$ for every $\phi \in M(A)$ implies $x=0$. Throughout this paper, the symbol C will denote the complex numbers.

LEMMA 1. *Let A be a commutative algebra and $\phi_1, \phi_2, \cdots, \phi_n$ be distinct homomorphisms of A onto C. There is an x in A such that $\phi_i(x)=0$ for $i=1, 2, \cdots, n-1$ and $\phi_n(x)=1$.*

PROOF. Since $\phi_i \neq \phi_n$ for $i \neq n$, we can find x_i in A such that $\phi_i(x_i)=0$ and $\phi_n(x_i)=1$. Let $x=\prod_{i=1}^{n-1} x_i$.

The following lemma appeared in [4] for a different type of topological algebra.

LEMMA 2. *Let A be a commutative F-algebra. Let $\phi_1, \phi_2, \cdots$ be a sequence of distinct points from $M(A)$ and let k be a positive integer. Then there is a y_k in A such that $\phi_i(y_k)=0$ for $1 \leqq i < k$ and $\phi_i(y_k) \neq 0$ for $i \geqq k$.*

PROOF. Use Lemma 1 to find $x_j \in A$, $j=1, 2, \cdots$, such that $\phi_i(x_j)=0$ for $i=1, 2, \cdots, j-1$ and $\phi_j(x_j)=1$. Set $y_k=\sum_{j=k}^{\infty} E_j x_j$ where the E_j are defined inductively. We define $E_k, E_{k+1}, \cdots$ by $E_k=1$ and, for $j>k$, $E_j=0$ if $\phi_j(\sum_{i=k}^{j-1} E_i x_i) \neq 0$; and if $\phi_j(\sum_{i=k}^{j-1} E_i x_i)=0$ we choose E_j such that $0<E_j$ and $\|E_j x_j\|_j < 2^{-j}$. Here $\|\cdot\|_j$ denotes the jth seminorm on A and we assume the seminorms are increasing. Since $\|E_j x_j\|_j < 2^{-j}$ and the seminorms are increasing, we have that $\sum_{j=k}^{\infty} E_j x_j$ converges to an element y_k of A. It is clear from the construction that y_k has the desired properties.

The next lemma appeared in [7]. We sketch a proof and refer the reader to [7] for the details.

LEMMA 3. *Let A be a commutative F-algebra. A can be realized as the inverse limit of a sequence of Banach algebras A_n. Let $\phi \in M(A)$. If ϕ is isolated in each of the spectra $M(A_n)$ which contains it, then ϕ is isolated in $M(A)$.*

PROOF. Use the Šilov idempotent theorem to obtain an idempotent $e_n \in A_n$ such that $\phi(e_n)=1$ and $\psi(e_n)=0$ for any ψ in $M(A_n)-\{\phi\}$. The idempotents e_n define an idempotent e in A such that $\phi(e)=1$ and $\psi(e)=0$ for any ψ in $M(A)-\{\phi\}$.

THEOREM 4. *Let B be an F-algebra and A be a commutative semi-simple F-algebra such that $M(A)$ has no isolated points. If Ψ is a homomorphism of B onto A, then Ψ is continuous.*

PROOF. The closed graph theorem is valid for F-algebras [**2**, p. 57]. Hence if we can show the graph of Ψ is closed, then we will be finished.

Let x_n be a sequence in B such that $x_n \to x$ in B and $\Psi(x_n) \to y$ in A. Let $S = \{\phi \in M(A) : \phi\Psi$ is a continuous homomorphism of B onto the complex numbers$\}$. Then for $\phi \in S$, $(\phi\Psi)(x_n) \to (\phi\Psi)(x)$ and $\phi(\Psi(x_n)) \to \phi(y)$, hence $\phi(\Psi(x)) = \phi(y)$. Therefore if we can show S separates the points of A, then we will be able to conclude that the graph of Ψ is closed. We will show that S separates the points of A by proving that S is dense in $M(A)$.

Assume there is a point ϕ in $M(A)$ which is not in the closure of S. Let A be the inverse limit of a sequence of Banach algebras A_n. Since ϕ is not isolated in $M(A)$, Lemma 3 implies there is an integer n such that ϕ is not isolated in $M(A_n)$. Since the Gelfand topology is T_2 and ϕ is not in the closure of S, we have that $M(A_n)$ must contain infinitely many points which are not in S. Choose a sequence $\phi_1, \phi_2, \cdots$ of distinct points from $M(A_n) - S$.

Use Lemma 2 to obtain a sequence $y_1, y_2 \cdots$ of points from A such that $\phi_i(y_k) = 0$ for $i < k$ and $\phi_i(y_k) \neq 0$ for $i \geqq k$. Let $z_1, z_2, \cdots$ be elements of B such that $\Psi(z_i) = y_i$ for $i = 1, 2, \cdots$. Use induction to construct a sequence $x_1, x_2, \cdots$ in B such that

(i) $\max_{1 \leq j \leq i} ||| z_j \cdots z_i x_i |||_i < 2^{-i}$, and

(ii) $|\phi_i\Psi(x_i)| > (|\phi_i\Psi \sum_{j=1}^{i-1} z_1 \cdots z_j x_j| + i)|\phi_i\Psi(z_1 \cdots z_i)|^{-1}$.

Here $||| \cdot |||_i$ denotes the ith seminorm on B and the seminorms on B are assumed to be increasing. It is always possible to choose x_i satisfying (i) and (ii) since $\phi_i\Psi$ is not continuous at zero.

Let $x = \sum_{i=1}^{\infty} z_1 \cdots z_i x_i$. Condition (i) and the fact that the seminorms are increasing guarantee that the sum converges.

For each positive integer $k > 1$, we have

$$\begin{aligned} \phi_k\Psi(x) &= \phi_k\Psi\left(\sum_{i=1}^{k-1} z_1 \cdots z_i x_i\right) + \phi_k\Psi(z_1 \cdots z_k x_k) \\ &\quad + \phi_k\Psi(z_1 \cdots z_{k+1} x_{k+1}) \\ &\quad + \phi_k\Psi\left[(z_1 \cdots z_{k+1}) \sum_{i=k+2}^{\infty} z_{k+2} \cdots z_i x_i\right] \\ &= \phi_k\Psi\left(\sum_{i=1}^{k-1} z_1 \cdots z_i x_i\right) + \phi_k\Psi(z_1 \cdots z_k x_k). \end{aligned}$$

Therefore

$$|\phi_k\Psi(x)| > |\phi_k\Psi(z_1 \cdots z_k x_k)| - \left|\phi_k\Psi\left(\sum_{i=1}^{k-1} z_1 \cdots z_i x_i\right)\right| > k.$$

This is impossible since $\phi_k \in M(A_n)$ for each k and therefore $|\phi_k(\Psi(x))| \leqq \|\Psi(x)\|_n$ where $\|\cdot\|_n$ is the seminorm on A corresponding to the Banach algebra A_n. This contradiction implies that S is dense in $M(A)$.

Since S is dense in $M(A)$ and A is semisimple, S must separate the points of A. As noted earlier, knowing that S separates the points of A allows us to conclude that Ψ is continuous.

REMARK. If we could remove the requirement that $M(A)$ has no isolated points from the hypotheses of Theorem 4, then we would have as a special case of Theorem 4 that every homomorphism of B onto the complex numbers is continuous.

THEOREM 5. *Let A be a semisimple commutative F-algebra with respect to a topology τ then τ is the only topology with respect to which A is an F-algebra.*

PROOF. Assume A is an F-algebra with respect to a second topology τ_1. Let i be the identity map from (A, τ_1) to (A, τ). We will use the closed graph theorem to show that i is continuous. It will then follow from the open mapping theorem that i must be a homeomorphism.

Let $S = \{\phi \in M(A, \tau) : \phi \in M(A, \tau_1)\}$. We will show that S is dense in $M(A, \tau)$. If S is dense in $M(A, \tau)$, then S separates the points of A. And if S separates the points of A, then the graph of i is closed.

Let ϕ be a point in $M(A, \tau)$. If ϕ is not isolated in $M(A, \tau)$, then we can show as in the proof of Theorem 4 that ϕ is in the closure of S.

Assume that ϕ is isolated in $M(A, \tau)$. The Šilov idempotent theorem implies there is an idempotent e in A such that $\phi(e) = 1$ and $\psi(e) = 0$ for ψ in $M(A, \tau) - \{\phi\}$. We do not know that ϕ is continuous on (A, τ_1); however, there is an element ϕ' of $M(A, \tau_1)$ such that $\phi'(e) = 1$ [**5**, p. 25]. Let x be an element of A, then for any ψ in $M(A, \tau)$ we have $\psi(xe - \phi(x)e) = 0$. Since A is semisimple, we have $xe = \phi(x)e$. For any x in A, $\phi'(x) = \phi'(xe) = \phi'(\phi(x)e) = \phi(x)$. Hence $\phi' = \phi$ which implies ϕ is continuous with respect to τ_1. Therefore ϕ is in S.

We have shown that S is dense in $M(A, \tau)$. We conclude that i must be a homeomorphism.

One of the most intriguing questions which remains open for F-algebras is the question of whether all homomorphisms of a commutative F-algebra onto the complex numbers are necessarily continuous.

It is shown in [5] that for purposes of this question it is sufficient to consider semisimple commutative F-algebras. We have the following corollary to Theorem 5.

COROLLARY 6. *Let A be a commutative semisimple F-algebra and ϕ be a homomorphism of A onto C. If there is an F-algebra topology for A with respect to which ϕ is continuous, then ϕ is already continuous on A.*

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