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HOMOGENEOUS INVERSE LIMIT SPACES WITH NONREGULAR COVERING MAPS AS BONDING MAPS

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ABSTRACT. We construct counterexamples to the conjecture that, in an inverse sequence (X, f) of closed manifolds X_n with covering maps $f_m^n:X_n\to X_m$ as bonding maps, if the inverse limit space is homogeneous, then there exists an integer m such that (for all $n>m$) the covering map $f_m^n:X_n\to X_m$ is regular.

1. **Introduction.** In 1966 R. M. Schori [5] conjectured that, in an inverse limit sequence (X, f) of closed manifolds X_n with covering maps $f_m^n:X_n\to X_m$ as bonding maps, if the inverse limit space is homogeneous, then there exists an integer m such that (for all $n>m$) the covering map $f_m^n:X_n\to X_m$ is regular. We present counterexamples to this conjecture in all dimensions greater than one.

This conjecture is the converse to McCord's theorem [2] that if (X, f) is an inverse limit sequence of closed manifolds and each bonding map $f_1^n:X_n\to X_1$ is a regular covering map, then the limit space is homogeneous. Examples of inverse limit sequences of closed manifolds with covering maps as bonding maps where the limit spaces fail to be homogeneous have been given in [3] and [5].

2. **Preliminaries and notation.** We let (X, f) denote the inverse limit sequence with factor spaces X_n and bonding maps $f_m^n:X_n\to X_m$ $(n\geqq m\geqq 1)$. The limit space, $\lim(X, f)$, is denoted by X_∞. If each factor space X_n is a closed, connected manifold and each bonding map f_m^n a covering map, then we refer to (X, f) as a *weak solenoidal sequence* and to X_∞ as a *weak solenoidal space*. We follow the notation in [1] for inverse limits. The reader is referred to [4] for the theory of covering spaces.

LEMMA 1. *Let A, B, and C be manifolds and let $f:A\to B$ and $g:B\to C$ be covering maps. If the composite $g\circ f:A\to C$ is a regular covering map, then f is also a regular covering map.*

We assume throughout that basepoints have been chosen nicely and are fixed.

LEMMA 2. *In the commutative diagram*

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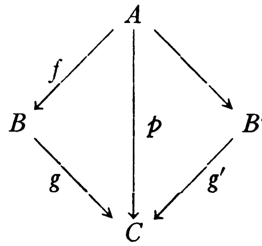
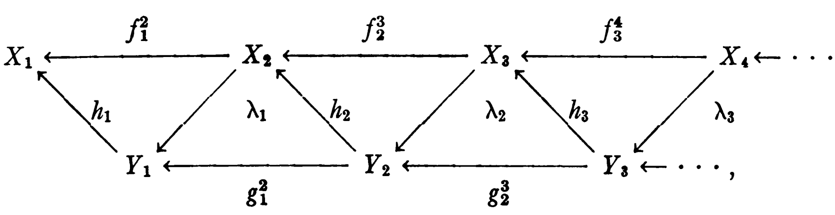


of manifolds and finite-to-one regular covering maps, let $p_*\pi_1(A) = (g_*\pi_1(B)) \cap (g'_*\pi_1(B'))$. *Moreover, suppose the multiplicity* α *of* g *is relatively prime to the multiplicity* α' *of* g'. *Then* g' *has the property that it can be factored into the composition of the covering maps* $B' \xrightarrow{h} D \xrightarrow{h'} C$ *such that* h' *is not regular if and only if* f *also has this property.*

PROOF. The factorization of g' into $h' \circ h$ is equivalent to the existence of a nonnormal subgroup of $\pi_1(C)$ that contains $g'_*\pi_1(B')$, namely $h'_*\pi_1(D)$. It is sufficient to observe that $\pi_1(B)/f_*\pi_1(A)$ is isomorphic to $\pi_1(C)/g'_*\pi_1(B')$, since this implies that a nonnormal subgroup containing $f_*\pi_1(A)$ occurs in $\pi_1(B)$ if and only if one containing $g'_*\pi_1(B')$ occurs in $\pi_1(C)$. But the indices α and α' of $g_*\pi_1(B)$ and $g'_*\pi_1(B')$, respectively, in $\pi_1(C)$ are relatively prime and so $\pi_1(C)$ is the smallest subgroup of itself containing both. Hence the above isomorphism follows from the isomorphism theorem.□

3. **Equivalent sequences.** Let (X, f) be a weak solenoidal sequence in which each bonding map $f_1^n: X_n \to X_1$ is a regular covering map. The limit space X_∞ is homogeneous [2]. Let us assume that each map $f_n^{n+1}: X_{n+1} \to X_n$ can be factored as the composition of a regular covering map $\lambda_n: X_{n+1} \to Y_n$ and a nonregular covering map $h_n: Y_n \to X_n$. We have the commutative diagram



where we define $g_n^{n+1} = \lambda_n \circ h_{n+1}$. The bottom row determines a weak solenoidal sequence (Y, g) with the property that no bonding map g_m^n $(n > m)$ is regular (see Lemma 1). However, $Y_\infty = \lim(Y, g)$ is clearly homeomorphic to X_∞ and is thus homogeneous.

In the next section we construct weak solenoidal sequences (X, f) that can be factored in this manner.

4. **Construction of** (X, f). Let G be the group with presentation $(a, b: a^\alpha = 1, b^{\alpha^2} = 1, a^{-1}ba = b^{\alpha+1})$, where α is a positive prime integer. G_α is a nonhamiltonian group of order α^3. Let X_1 be a closed n-manifold such that there is an epimorphism $\eta_\alpha: \pi_1(X_1) \to G_\alpha$ for every prime α. For example, we could let X_1 be the n-manifold $(n \geqq 2)$ obtained from two triangulated copies of $S^1 \times S^{n-1}$ by removing the interior of a polyhedral n-cell from each and then matching the resulting $(n-1)$-sphere boundaries by a piecewise linear homeomorphism (this is the *connected sum*, denoted by $(S^1 \times S^{n-1}) \# (S^1 \times S^{n-1})$). Then $\pi_1(X_1) \cong Z_* Z$ if $n > 2$, and $\pi_1(X_1) \cong (a, b, c, d: [a, b][c, d] = 1)$ if $n = 2$.

Let $\alpha_2 < \alpha_3 < \cdots$ be a fixed sequence of prime integers. For each $i \geqq 2$ define $p_i: A_i \to X_1$ to be the covering map such that $(p_i)_* \pi_1(A_i)$ = the kernel of η_{α_i}. Then p_i is a regular covering map of multiplicity α_i^3 and factors into the composition of a regular and a nonregular covering map (since $\pi_1(X_1)/(p_i)_* \pi_1(A_i) \cong G_{\alpha_i}$ has a nonnormal subgroup). Since p_i has a finite multiplicity, A_i is a closed manifold.

We define the sequence (X, f) inductively. Suppose then that X_1 has already been fixed as described in the preceding paragraph. Define $X_2 = A_2$ and $f_1^2: X_2 \to X_1$ to be p_2. Now suppose we have defined X_i and $f_{i-1}^i: X_i \to X_{i-1}$ (for $i \leqq k$) such that each $f_1^i: X_i \to X_1$ is a regular covering map of multiplicity $\alpha_2 \alpha_3 \cdots \alpha_i$ and each f_{i-1}^i factors into a regular and a nonregular covering map. To define $f_k^{k+1}: X_{k+1} \to X_k$ consider the covering maps $f_1^k: X_k \to X_1$ and $p_{k+1}: A_{k+1} \to X_1$. Define $f_1^{k+1}: X_{k+1} \to X_1$ to be the covering map such that

$$(f_1^{k+1})_* \pi_1(X_{k+1}) = [(f_1^k)_* \pi_1(X_k)] \cap [(p_{k+1})_* \pi_1(A_{k+1})].$$

f_1^{k+1} can be factored through f_1^k, so define f_k^{k+1} such that $f_1^{k+1} = f_1^k \circ f_k^{k+1}$. Clearly f_1^{k+1} is a regular covering map. Furthermore, by Lemma 2, the regular covering map f_k^{k+1} can be factored into a regular and a nonregular covering map since p_{k+1} has this property. This defines the weak solenoidal sequence (X, f) that produces the desired counterexample.

5. **Questions.** Let (X, f) and (Y, g) be weak solenoidal sequences. We say that (X, f) is *equivalent* to (Y, g) if there is a third weak solenoidal sequence in which subsequences of both (X, f) and (Y, g) can be embedded. For example, the sequences (X, f) and (Y, g) of §3 are equivalent.

QUESTION 1. If $X_\infty = \lim(X, f)$ is a homogeneous weak solenoidal space, is (X, f) equivalent to another weak solenoidal sequence (Y, g) in which each of the bonding maps g_1^n are regular covering maps?

QUESTION 2. Let X_∞ be a weak solenoidal space in which all the path components are homeomorphic. Is X_∞ homogeneous?

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