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FIELDS WITH FEW EXTENSIONS

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ABSTRACT. We show that a valued field Λ with only a finite number of nonisomorphic valued extensions is equal to the complex field C or is real closed with $C=\Lambda(\sqrt{(-1)})$.

The Ostrowski (Gelfand-Mazur) Theorem [2, p. 131], [4, p. 260] implies that with any of the valuations $v(x)=|x|^t$, where $|\cdot|$ denotes the usual modulus and $0<t\leqq 1$, the real field R has essentially only one proper valued field extension, the complex field C, and the complex field has no proper valued field extension. We investigate *which valued fields* Λ *have only a finite number of nonisomorphic valued field extensions.* We take all valuations to be subadditive maps into R, and (unless specifically stated otherwise) all *isomorphisms* of [valued] extensions of a [valued] field Λ to be [isometric] Λ-algebra isomorphisms.

Before considering the case of valued fields, we consider a purely algebraic case by restricting to finite dimensional extensions. A field with no proper finite dimensional extensions is algebraically closed, since an extension of a field by a single algebraic element is a finite dimensional extension. On the other hand, it is well known that a real closed field has essentially just one proper finite extension, where a *real closed* field is a formally *real* field Λ (that is, no sum of squares of nonzero elements in Λ is zero) such that no proper algebraic extension of Λ is formally real [4], [5].

1 PROPOSITION. *A field* Λ *has only a finite number of finite dimensional nonisomorphic extensions if, and only if,* Λ *is algebraically closed or real closed.*

PROOF. Suppose that there are only a finite number of finite dimensional nonisomorphic extensions of Λ. Let Φ be an extension of maximal finite degree over Λ. If Φ is not algebraically closed, then there is a prime polynomial p of degree greater than 1 over Λ. Then $\Phi'=\Phi[x]/(p)$ is a proper finite extension of Φ, and hence a finite extension of Λ of degree greater than the degree of Φ. Since this is impossible, Φ must be algebraically closed. If $\Phi\neq\Lambda$, the Artin-Schreier The-

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orem [4, p. 316] implies that Λ is real closed (and $\Phi=\Lambda(\sqrt{(-1)})$). The proposition follows.

Now recall that the only archimedean valuations on the complex field are of the form $v_t(x)=|x|^t$ for all x in $\boldsymbol{C}$ where $0<t\leqq 1$.

2 Proposition. *A valued field* (Λ, v) *has only a finite number of nonisomorphic valued field extensions if, and only if,* $(\Lambda, v)\cong(\boldsymbol{C}, v_t)$ *or* Λ *is real closed with* $\Lambda(\sqrt{(-1)})\cong\boldsymbol{C}$ *and* v *corresponds to a restriction of* v_t *for some* t $(0<t\leqq 1)$.

Proof. Suppose that the second condition is satisfied: if $(\Lambda, v)\cong(\boldsymbol{C}, v_t)$ or $(\boldsymbol{R}, v_t)$, then, by the Gelfand-Mazur Theorem [2, p. 127], the only valued extensions of Λ are essentially $(\boldsymbol{C}, v_t)$ or $(\boldsymbol{C}, v_t)$, $(\boldsymbol{R}, v_t)$, respectively. If $(\Lambda(\sqrt{(-1)}), v)\cong(\boldsymbol{C}, v_t)$ and $(\Lambda, v)\not\cong(\boldsymbol{R}, v_t)$ or $(\boldsymbol{C}, v_t)$, let Φ be a proper valued extension of Λ. Then, since the valuation on Φ is archimedean, by Ostrowski's Theorem [2, p. 131], there is an isomorphism θ from Φ onto a dense subfield of $\boldsymbol{C}$ such that $v=v_r\theta$ for some r $(0<r\leqq 1)$. Now θ restricted to Λ induces an isometric isomorphism from a dense subfield of $(\boldsymbol{C}, v_t)$ (isomorphic to Λ and of codimension 2) into the dense subfield $\theta(\Lambda)$ of $(\boldsymbol{C}, v_r)$. Thus $r=t$ [2, p. 131]. Since $(\boldsymbol{C}, v_r)$ is complete, this isomorphism has an isometric extension ψ from $(\boldsymbol{C}, v_t)$ onto $(\boldsymbol{C}, v_t)$ [4, p. 221]. Since $\psi^{-1}\theta(\Lambda)$ has codimension 2 in $\boldsymbol{C}$, and $\theta(\Lambda)\neq\theta(\Phi)$, it follows that $\theta(\Phi)=\boldsymbol{C}$, that is $(\Phi, v)\cong(\boldsymbol{C}, v_t)$.

Conversely suppose that Λ has only a finite number of nonisomorphic valued field extensions. If the valuation v on Λ is nonarchimedean, it can be extended to any field extension of Λ [5, p. 299], and there are an infinite number of nonisomorphic field extensions of any field. Hence v is archimedean, so, by Ostrowski's Theorem [2, p. 131], there is an isomorphism θ from Λ onto a dense subfield of $\boldsymbol{R}$ or $\boldsymbol{C}$, and a t $(0<t\leqq 1)$ such that $v=v_t\theta$. We identify Λ with the subfield $\theta(\Lambda)$ of $\boldsymbol{C}$ with valuation v_t.

If B is a transcendency basis of $\boldsymbol{C}$ over Λ, then $\Lambda(B)$, the subfield of $\boldsymbol{C}$ generated by Λ and B, has only a finite number of nonisomorphic valued extensions, and $\boldsymbol{C}$ is algebraic over $\Lambda(B)$. Now a finite dimensional extension of a valued field has a valuation that extends the given valuation [5, p. 292], so that as an abstract field $\Lambda(B)$ has only a finite number of nonisomorphic finite dimensional extensions. By Proposition 1, $\Lambda(B)$ is algebraically closed or $\Lambda(B)$ is real closed with $\Lambda(B)(\sqrt{-1})$ algebraically closed; hence $\Lambda(B)=\boldsymbol{C}$ or $\Lambda(B)(\sqrt{(-1)})=\boldsymbol{C}$. If B is empty the proposition is proved, so we assume that B is not empty. Let x be an element of B, and let Φ be the subfield of

$\boldsymbol{C}$ generated by Λ, $B\setminus\{x\}$, and $\sqrt{(-1)}$ ($\sqrt{(-1)}$ may already be in Λ). Then $\boldsymbol{C}=\Phi(x)$ is a simple transcendental extension of Φ. Since $\boldsymbol{C}$ is algebraically closed there is a $\lambda=p(x)/q(x)$, where p and q are polynomials with coefficients in Φ, such that $\lambda^2-x=0$. Hence $p(x)^2-xq(x)^2=0$, so that, by the unique factorization of polynomials in a transcendental element, x is algebraic over Φ. This contradiction completes the proof.

3 Remarks. (i) There are archimedean real closed fields Λ contained in $\boldsymbol{C}$ with $\Lambda(\sqrt{(-1)})=\boldsymbol{C}$ and $\Lambda\cong\boldsymbol{R}$ as a field but ***not*** as a valued field. For, it is well known that there are infinitely many field automorphisms θ of $\boldsymbol{C}$ not leaving $\boldsymbol{R}$ invariant [4, p. 157, say], and $\Lambda=\theta(\boldsymbol{R})$ is a real field with $\Lambda(\sqrt{(-1)})=\boldsymbol{C}$. If $\Lambda\neq\boldsymbol{R}$ it cannot be topologically isomorphic to $\boldsymbol{R}$, since the only closed topological subfields of $\boldsymbol{C}$ are $\boldsymbol{R}$ and $\boldsymbol{C}$ [1, §3].

(ii) Since the only complete subfields of $(\boldsymbol{C}, v_t)$ are $\boldsymbol{C}$ and $\boldsymbol{R}$, Proposition 2 implies that the only *complete* valued fields with only a finite number of nonisomorphic valued field extensions are essentially $(\boldsymbol{C}, v_t)$ and $(\boldsymbol{R}, v_t)$ $(0<t\leqq 1)$.

(iii) Another corollary is that a valued field has ***no*** proper valued field extension if, and only if, it is isomorphic to $(\boldsymbol{C}, v_t)$ for some t $(0<t\leqq 1)$.

(iv) This last result shows that the only valued field over which the whole Gelfand theory of commutative Banach algebras can be developed is the complex field (see, for example, [6], [7]).

(v) By the Artin-Schreier Theorem [4, p. 316], every proper subfield Λ of finite codimension in $\boldsymbol{C}$ is real closed with $\Lambda(\sqrt{(-1)})=\boldsymbol{C}$. With reference to remark (i) above, it would be interesting to know whether or not every subfield of codimension 2 in $\boldsymbol{C}$ is field isomorphic to $\boldsymbol{R}$.

(vi) In the second half of the proof of Proposition 2, after identifying Λ with a subfield of $\boldsymbol{C}$, one may show that the transcendency degree of $\boldsymbol{C}$ over Λ is finite, and hence that $\boldsymbol{C}$ is the extension of Λ by a finite number of elements. Then a result of E. Fried [3] implies that $\Lambda=\boldsymbol{C}$ or Λ is real closed with $\Lambda(\sqrt{(-1)})=\boldsymbol{C}$.

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