A HOMOLOGICAL CHARACTERIZATION OF STEINITZ RINGS

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ABSTRACT. A ring R (associative with an identity) is called a right Steinitz ring if any linearly independent subset of a free right R -module F can be extended to a basis of F . Steinitz rings have been investigated in a recent paper of Chwe and Neggers. In this note it is shown that the right Steinitz rings are exactly the right perfect, local rings.

1. Preliminaries. R denotes an associative ring with an identity, and all modules are unitary right R -modules. Obviously R is a right Steinitz ring iff, for any free right R -module F , and any free submodule U of F , F/U is again free (and U a direct summand of F). Every projective R -module P over a right Steinitz ring R is free: Let P be a direct summand of a free module F of infinite rank, then $F \cong P \oplus F$ by a standard argument. Therefore $P \cong P \oplus F/F$ is free. Consequently R is a right Steinitz ring iff any right R -module of finite homological dimension is free.

The Jacobson radical of R is denoted $J = J(R)$. R is local if R/J is a division ring. The right finitistic projective dimension $\text{rFPD}(R)$ is defined in the sense of H. Bass [1, p. 478].

2. Results. The following theorem characterizes right Steinitz rings.

THEOREM. *For any ring R the following statements are equivalent:*

- (a) *Any linearly independent subset of a free right R -module F can be extended to a basis of F by adjoining elements of a given basis.*
- (b) *R is a right Steinitz ring.*
- (c) *Any countable linearly independent subset of a (countably generated) free right R -module F can be extended to a basis of F .*
- (d) *R is right perfect and any finitely generated projective right R -module is free.*
- (e) *R is a right perfect, local ring.*

PROOF. We will give a cyclic proof. (a) $\Rightarrow$ (b) and (b) $\Rightarrow$ (c): Obvious. (c) $\Rightarrow$ (d): Following Bass [1], for any sequence $\{a_n\}_{n=0}^{\infty}$ of elements of

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R we consider a free right R -module F with basis $\{x_n\}_{n=0}^\infty$ and the submodule G of F generated by $\{x_n - x_{n+1}a_n\}_{n=0}^\infty$. Since G is free, G is a direct summand of F . Therefore the chain $\{Ra_n \cdots a_0\}_{n=0}^\infty$ of principal left ideals terminates [1, Lemma 1.3]. Thus, R is right perfect. By the argument preceding the theorem any countably generated projective right R -module is free.

(d) $\Rightarrow$ (e): We shall now prove that R/J is a division ring. Suppose N is any right ideal of R/J . Since R/J is semisimple artinian, N is a direct summand of R/J . J is a nil ideal; therefore idempotents may be lifted modulo J , and N has the form D/DJ with D a direct summand of R . Since D is free, N is a free right ideal of the artinian ring R/J . Thus, $N=0$ or $N=R/J$.

(e) $\Rightarrow$ (a): Since R is right perfect, the right annihilator of J is non-zero [2, Corollary 3]. Since R is local, any proper left ideal of R is contained in J , consequently has nonzero right annihilator. Thus $\text{rFPD}(R)=0$ [1, Theorem 6.3].

For any R -module E the R/J -module E/EJ is denoted by $\bar{E}$; for $x \in E$ the corresponding element of $\bar{E}$ is $\bar{x}$. Suppose now that B is a basis of the R -module F and S a linearly independent subset of F . Since $\text{rFPD}(R)=0$, the submodule G generated by S is a direct summand of F . Therefore $\bar{S} = \{\bar{s} : s \in S\}$ is a linearly independent subset of $\bar{F}$. Since R/J is a division ring and $\bar{B}$ a basis of $\bar{F}$, there is a subset T of B ($T \cap S = \emptyset$) such that $\bar{S} \cup \bar{T}$ is a basis of $\bar{F}$. The proof of (e) $\Rightarrow$ (a) is completed by the following lemma.

LEMMA. *Suppose R is right perfect, F is a projective right R -module and B is a subset of F . B is a basis of F iff $\bar{B}$ is a basis of the R/J -module $\bar{F} = F/FJ$.*

PROOF. Suppose $\bar{B}$ is a basis of $\bar{F}$. Let G be the submodule of F generated by B . Clearly $\bar{G} = \bar{F}$; i.e. $(F/G)J = F/G$. Therefore $F=G$ since J is right T -nilpotent [1, p. 473]. Now, if H is a free right R -module with basis B , the identity map on B induces an epimorphism $u: H \rightarrow F$ with kernel N . Since $0 \rightarrow N \rightarrow H \rightarrow F \rightarrow 0$ splits, the corresponding sequence $0 \rightarrow \bar{N} \rightarrow \bar{H} \rightarrow \bar{F} \rightarrow 0$ is exact. $\bar{u}: \bar{H} \rightarrow \bar{F}$ is an isomorphism by hypothesis, therefore $\bar{N}=0$. Again, since J is right T -nilpotent, $N=0$. Thus, u is an isomorphism and B is a basis of F .

3. Remarks. There is another characterization of right Steinitz rings. Govorov [3] proved that R is a right perfect, local ring iff every flat right R -module is free.

One may consider rings R with the Steinitz property for finite sets, i.e. every finite linearly independent subset of a (finitely generated)

free right R -module F may be extended to a basis of F . A ring with this property need not be a right Steinitz ring. In fact, any semisimple artinian ring has the Steinitz property for finite sets.

On the other hand, a right perfect ring R need not have the Steinitz property for finite sets. Otherwise, it would follow from [1, Theorem 5.4 and Theorem 6.3] that $\text{rFPD}(R) = 0$. The ring R of all (upper) triangular 2×2 -matrices over a field is right perfect and has $\text{rFPD}(R) = 1$.

H. Bass constructed a local ring R which is right perfect, but not left perfect [1, p. 476]. Therefore, a right Steinitz ring need not be a left Steinitz ring.

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