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DIRECT PRODUCT OF DIVISION RINGS AND A PAPER OF ABIAN

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ABSTRACT. It is shown that the rings under the title admit an order-theoretical characterization as in the commutative case studied by Abian.

Introduction. Let R be an associative ring equipped with the binary relation ($\leqq$) defined by $x \leqq y$ if and only if $xy = x^2$ in R. In this paper, it is shown that ($\leqq$) is an order relation on R if and only if, R has no nilpotent elements ($\neq 0$). Conditions on the binary relation ($\leqq$) in order that R split into a direct product of division rings are then studied in the light of Abian's result [1, Theorem 1]. Using similar argumentation and using certain subdirect representation of rings with no nilpotent elements, one obtains a complete similarity with the commutative case (yet, no extra complication in the computations).

CONVENTIONS. R is an associative ring which is, unless otherwise stated, with no nilpotent elements (other than 0). As a result of [2], R can be embedded into a direct product of skewdomains, $R \to \prod_{\iota \in I} R_\iota$ (that is to say, rings R_ι having no one-sided divisors of zero). The former embedding is fixed throughout the paper. It is therefore legitimate to identify any element x in R with the family consisting of all its projections $(x_\iota)_{\iota \in I}$. Finally, all definitions in [1] are extended (verbatim) to the present case (of a noncommutative ring R) and are freely used throughout.

In this paper we offer the following generalization to the noncommutative case of Abian's result [1, Theorem 1].

THEOREM. *Any ring R equipped with its binary relation ($\leqq$) defined by $a \leqq b$, if and only if $ab = a^2$, is isomorphic to a direct product of division rings if and only if ($\leqq$) is an order relation on R such that R is hyperatomic and orthogonally complete (in the sense of Abian).*

The 'only if' is just a combination of the forthcoming Lemma 2

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and a partial duplication of Abian's proof. (See first part of the proof of [**1**, Theorem 1].) The 'if' breaks into several lemmas some which are of independent interest.

LEMMA 1. *If* $a \leqq b$ *is an order relation on a ring* R, *then* R *has no nilpotent elements* $(\neq 0)$.

PROOF. Assume $x^2=0$ in R. Then $x \leqq 0$. However 0 is the least element under $\leqq$. Therefore $x=0$.

LEMMA 2. *If* R *has no nilpotent elements, then* $(\leqq)$ *is a (multiplicatively) permissible (that is to say,* $a \leqq b$ *implies* $ac \leqq bc$ *and* $ca \leqq cb$ *for all* $c \in R$*) order relation on* R.

PROOF. Using the mentioned embedding of R (see Conventions), it suffices to show the result for a skewdomain. In this case the occurrence $ab=a^2$, is equivalent with $a=0$ or $a=b$. Now the latter occurrence defines a binary relation which is obviously a permissible order relation on R. Lemma 2 is established.

Some known properties of rings without nilpotents R which follow at once from the considered embedding of R, are collected without proof in the next lemma.

LEMMA 3. *If* $(\leqq)$ *orders* R, *then*

(1) *Any idempotent element* e $(=e^2)$ *of* R *is in the center of* R, *and* $ex=xe \leqq x$ *for all* $x \in R$.

(2) *For any* $a, x \in R$ (i) $x^2a=x^2$ *implies* $xa=x$; (ii) $x^2a=x$ *implies* $xax=x$; (iii) $x^2a=0$ *implies* $xa=0$.

We assume henceforth that R is ordered, or equivalently (Lemmas 1 and 2), that R has no nilpotent elements. Along the lines of the proof of [**1**, Theorem 1] let us now show

LEMMA 4. *Let* a *be an hyperatom of* R. *For any* r *such that* $ar \neq 0$, ar *is an hyperatom.*

PROOF. Let $x \leqq ar$ in R. By definition [**1**, Definition 1, (6)], there is s so that $xs \leqq ars=a$. As $x \leqq ar$, $x^2=xar$ implies $x^2s=xars=xa$, and $x^2=(xa)r=(x^2s)r=x^2(sr)$. Then (Lemma 3, (2)(i)) $x=x(sr)=(xs)r$. As a is an hyperatom, either $xs=0$ and so, $x=(xs)r=0$, or $xs=a$ in which case $x=ar$. Finally, assume $(ar)y \neq 0$ for some given y. Then, for $t=t'r$ with t' so that $a(ry)t'=a$, we get $(ar)yt=a(ry)t'r=ar$ proving thereby that ar is an hyperatom.

LEMMA 5. *Let* $x \neq 0$ *in* R. *Assume that* $q \leqq x$ *for some hyperatom* $q \neq 0$. *Then there is an idempotent hyperatom* e *such that* $ex \neq 0$.

PROOF. As $q \neq 0$, $q^2 \neq 0$. Then $q^2 s = q$ for some s. It follows (Lemma 3, (2)(iii)) that $qsq = q$. Set $e = qs = sq$. As $q \neq 0$, $e \neq 0$ and by Lemma 4, e is an idempotent hyperatom. Evaluating ex, we get $ex = qsx = sqx = sq^2 = qsq = q \neq 0$.

LEMMA 6. *The set* $E = \{e_\iota, \iota \in I\}$ *of all idempotent hyperatoms of* R *is an orthogonal set and each of its elements* e_ι *generates a division ring* $D_\iota = e_\iota R$.

PROOF (SKETCHED). By definition of an hyperatom e such that $e = e^2$, and by property (1) in Lemma 3.

LEMMA 7. *If* R *is hyperatomic, then* $f = x \to (e_\iota x)_{\iota \in I}$ *is a monomorphism from* R *into a direct product of division rings* $D_\iota = e_\iota R$.

PROOF. It is an immediate consequence of Lemmas 5 and 6.

LEMMA 8. *The embedding in Lemma* 7 *has the following properties*:

(1) *Each factor* $D_\iota = e_\iota R$ *is a skewdomain.*

(2) *If* $(a^{(\alpha)})_{\alpha \in A}$ *is any family of elements of* R *having a supremum* a *in* R *then for any fixed* $\iota \in I$, $a_\iota^{(\alpha)} = 0$ *for all* $\alpha \in A$ *implies* $a_\iota = 0$.

PROOF. Let $(a^{(\alpha)})_{\alpha \in A}$ be a family of elements in R admitting supremum a in R. Let $\iota \in I$ such that $a_\iota^{(\alpha)} = 0$ for all $\alpha \in A$. To prove that $a_\iota = 0$. Here $a_\iota = e_\iota a$. Set $a' = a - e_\iota a$. As E is orthogonal,

$$\begin{aligned} a'_\mu &= a_\mu \quad \text{if } \mu \neq \iota, \\ &= 0 \quad \text{if } \mu = \iota. \end{aligned}$$

Then $a_\mu^{(\alpha)} \leqq a'_\mu$ for all $\mu \in I$. Consequently, $a^{(\alpha)} \leqq a'$ for all $\alpha \in A$. Then $a \leqq a'$, that is to say, $a(a - e_\iota a) = a^2$, if and only if, $a^2 e_\iota = 0$, equivalently $ae_\iota = 0$, equivalently, $a_\iota = 0$.

LEMMA 9. *If* R *admits at least one imbedding as in Lemma* 8, *then* R *has the following property*:

(A) *For any family* $(a^{(\alpha)})_{\alpha \in A}$ *of elements of* R *admitting a supremum in* R, *and any* $b \in R$, $(ba^{(\alpha)})_{\alpha \in A}$ *admits a supremum in* R *equal to* $b(\sup_\alpha a^{(\alpha)})$.

PROOF. As $(\leqq)$ is left permissible, $a^{(\alpha)} \leqq a$ implies $ba^{(\alpha)} \leqq ba$ for any $\alpha \in A$, and ba is an upper bound of $\{ba^{(\alpha)}, \alpha \in A\}$. Also, ba is the least upper bound. For let $ba^{(\alpha)} \leqq u$ in R. If $ba \leqq u$ were not true, then for some $\iota \in I$, $b_\iota a_\iota \neq u_\iota$. Then $a_\iota \neq 0$, and $a_\iota^{(\alpha_0)}$ for some $a_0 \in A$ follows. As $a_\iota^{(\alpha_0)} \leqq a_\iota$, in the skewdomain D_ι we must have $a_\iota^{(\alpha_0)} = a_\iota$ (see Lemma 2 and its proof). As $b_\iota a_\iota^{(\alpha)} \leqq u_\iota$ for all α, in particular, $b_\iota a_\iota$ $(= b_\iota a_\iota^{(\alpha_0)}) \leqq u_\iota$, a contradiction.

LEMMA 10. *Let R be an orthogonally complete ring satisfying* (A). *Let $F=\{e_\lambda, \lambda\in\Lambda\}$ be an orthogonal set of idempotents in R such that $\phi = x\to(e_\lambda x)_{\lambda\in\Lambda}$ is a monomorphism. Then ϕ is an isomorphism.*

PROOF. For let $x^{(\lambda)}\in e_\lambda R$, λ ranging over Λ. As $X=\{x^{(\lambda)}, \lambda\in\Lambda\}$ is orthogonal it admits a supremum x in R. Evaluating the λth projection of x on $e_\lambda R$ we get

$$x_\lambda = e_\lambda x = e_\lambda \left(\sup_{\mu\in\Lambda} x^{(\mu)}\right) = \sup_\mu (e_\lambda x^{(\mu)}) = \sup\{0, e_\lambda x^{(\lambda)}\} = e_\lambda x^{(\lambda)} = x^{(\lambda)}$$

for all $\lambda\in\Lambda$. Thus ϕ is epi. The lemma is established and implies immediately together with Lemmas 7, 8 and 9 the required splitting of R.

REMARK. As shown by Abian, Property (A) holds for any commutative ring R without nilpotent elements [**1**, Lemma 8]. Note also that Abian's proof of (A) does not use subdirect representation of R.

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