A NOTE ON MODULES OVER A COMMUTATIVE REGULAR RING

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ABSTRACT. An example is given of a commutative, von Neumann regular ring R , which has a module A satisfying the following conditions: (1) $T(A) = \{a \in A \mid (0:a) \text{ is an essential ideal of } R\}$ is a cyclic R -module; (2) $A/T(A)$ is a cyclic R -module; and (3) $T(A)$ is not a direct summand of A . This answers in the negative a question raised by R. S. Pierce.

In Pierce's memoir [2], a module M over a commutative regular ring R is called *torsion* if $\text{Hom}_R(M, E(R)) = 0$, where $E(R)$ denotes the injective hull of R . This definition for torsion modules over a commutative regular ring coincides with several other concepts of torsion for modules over more general rings (see [1]). The following question has been asked by Pierce [2, p. 109]: "If the torsion submodule $T(A)$ is finitely generated and $A/T(A)$ is a finite direct sum of cyclic modules, does it follow that $T(A)$ is a direct summand of A ?" The purpose of this note is to answer this question in the negative by giving an example of a module A such that $T(A)$ and $A/T(A)$ are cyclic, but $T(A)$ is not a summand of A .

Let B be a non-Artinian Boolean ring with identity 1. For each positive integer n , let $B^{(n)} \cong B$. Let $P = \prod_{n=1}^{\infty} B^{(n)}$, and define elements e and f of P as follows:

$$e = (1, 1, 1, \dots)$$

and

$$\begin{aligned} f &= (a_1, a_2, a_3, \dots), & \text{where } a_n &= 1 & \text{if } n = 2k, \quad k = 1, 2, \dots, \\ & & &= 1 & \text{if } n = 3^s, \quad s = 1, 2, \dots, \\ & & &= 0 & \text{otherwise.} \end{aligned}$$

Let R be the subring of P generated by e and $\bigoplus \sum_{n=1}^{\infty} B^{(n)}$. Since B is not Artinian, there exists an essential ideal I of B . Let $N_0 = \bigoplus \sum_{n=1}^{\infty} I^{(n)}$ and $N_1 = \bigoplus \sum_{n=1}^{\infty} I^{(2n)}$, where $I^{(n)} \cong I$ for each integer n . Let N be the R -submodule of $(Re/N_0) \oplus (Rf/N_1)$ generated by all elements of the form

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$$((0, v_2, 0, v_4, 0, v_6, \dots) + N_0, (0, v_2, 0, v_4, 0, v_6, \dots) + N_1).$$

Define $A = ((Re/N_0) \oplus (Rf/N_1))/N$.

Since a commutative regular ring R has zero singular ideal, it is well known (see [1]) that an element x of an R -module X is in $T(X)$ if and only if $(0:x)$ is an essential ideal of R . From this fact and the definition of N , it can be verified that $T(A)$ is generated by the element $(e + N_0, (0, 0, 0, \dots) + N_1) + N$.

Since $T(Re/N_0) = Re/N_0$, it follows from the closure properties (see [1] or [2]) of torsion classes that there is a naturally induced epimorphism from

$$Rf/N_1 \cong ((Re/N_0) \oplus (Rf/N_1))/(Re/N_0)$$

onto $A/T(A)$. Therefore $A/T(A)$, being the epimorphic image of a cyclic module, is cyclic.

Let $Ry \subseteq A$ such that $T(Ry) = 0$, and write

$$y = ((t_1, t_2, t_3, \dots) + N_0, (b_1, b_2, b_3, \dots) + N_1) + N.$$

Now let $h_k = (c_1, c_2, c_3, \dots)$, where $c_k = 1$ and $c_i = 0$ for all $i \neq k$. Thus

$$h_{t_i} y = ((0, 0, \dots, 0, t_i, 0, \dots) + N_0, (0, 0, \dots) + N_1) + N \in T(A)$$

for all $i = 1, 2, \dots$. Since $T(Ry) = 0$, it follows that $t_i \in I^{(t_i)}$ for all $i = 1, 2, \dots$. Hence there exists a positive integer k such that $t_i \in I^{(i)}$ for all $i \geq k$. Then

$$\begin{aligned} h_{2i} y &= ((0, 0, \dots, 0, t_{2i}, 0, \dots) \\ &\quad + N_0, (0, 0, \dots, 0, b_{2i}, 0, \dots) + N_1) + N \end{aligned}$$

is an element of $T(A)$; hence $b_{2i} \in I^{(2i)}$ for all $2i \geq k$. Thus there exists a positive integer $s \geq k$ such that $b_i \in I^{(i)}$ for all $i \geq s$. It is now easy to verify that

$$((0, 0, \dots) + N_0, h_{3^s} + N_1) + N \notin T(A) \oplus Ry.$$

If $T(A)$ were a direct summand of A , then by the fact that $A/T(A)$ is cyclic, there would exist a cyclic module $Ry \subseteq A$ such that $T(Ry) = 0$ and $A = T(A) \oplus Ry$. By the preceding paragraph, such a module Ry cannot exist; hence $T(A)$ cannot be a direct summand of A .

REFERENCES

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