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COMPLETENESS OF $\{\sin nx + Ki \cos nx\}$

JONATHAN I. GINSBERG[1]

ABSTRACT. Let $C[a, b]$ be the space of continuous functions on the interval $[a, b]$. It is shown that the set of functions $\{\sin nx + Ki \cos nx\}_{n=1}^{\infty}$, $K \neq \pm 1$, is incomplete in $C[0, \pi + a]$, $a > 0$.

In [1] it was proved that the set of functions $\{\sin nx + \lambda \cos nx;\ n = 1, 2, \cdots\}$ is complete in $C[0, \pi]$ if and only if $\lambda = Ki$ with $-1 \leqq K \leqq 1$ and $K \neq 0$. When $K = 1$ or $K = -1$, we have, respectively, the sets $\{ie^{-inx}\}$ or $\{-ie^{inx}\}$, each of which is complete in $C[0, 2\pi - \epsilon]$ for any $\epsilon > 0$ [2, p. 3]. If, on the other hand, $K = 0$, our set of functions is $\{\sin nx\}$, and this set is not complete in $C[0, \pi]$ even if we adjoin 1 to it, because only functions taking the same value at 0 and at π can be approximated by linear combinations of its members. The question arises as to whether the "largest" a permitting completeness of $\{\sin nx + Ki \cos nx\}$ in $C[0, \pi + a]$ might depend continuously on K for $-1 \leqq K \leqq 1$. The purpose of this note is to show that the answer to that question is negative.

THEOREM. *If $K \neq \pm 1$, the set of functions $\{\sin nx + Ki \cos nx;\ n = 1, 2, \cdots\}$ is not complete in $C[0, \pi + a]$ for any $a > 0$.*

PROOF. By the above cited result of [1], it is only necessary to consider the case when $-1 < K < 1$ and $K \neq 0$.

We have

$$\sin nx + Ki \cos nx = \frac{(K-1)i}{2}\left[e^{inx} - \frac{1+K}{1-K} e^{-inx}\right],$$

so, reasoning by duality, we can establish the theorem by producing a bounded function $f(\theta)$, not identically zero, with

$$(1) \qquad \int_0^{\pi + a} \left[e^{in\theta} - \frac{1+K}{1-K} e^{-in\theta}\right] f(\theta)\, d\theta = 0, \qquad n = 1, 2, \cdots.$$

Suppose $F(z)$ is analytic and bounded in $|z| < 1$, and continuous up

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to $|z|=1$ save perhaps at 1 and -1. Then, by elementary Fourier series theory,

$$\int_0^{2\pi}\left[F(e^{i\theta})+\frac{1+K}{1-K}F(e^{-i\theta})\right]\left[e^{in\theta}-\frac{1+K}{1-K}e^{-in\theta}\right]d\theta=0$$

for $n=1, 2, \cdots$. It will thus be enough to construct such a function F with

$$f(\theta)=F(e^{i\theta})+\frac{1+K}{1-K}F(e^{-i\theta})$$

vanishing identically on $[\pi+a, 2\pi]$ but not on $[0, 2\pi]$; (1) will then hold for this f.

The function $p(z)=-i(1+z)/(1-z)$ maps $|z|<1$ conformally onto the lower half plane, taking -1 to 0, i to 1, and 1 to ∞. If, then, $r<1$ is very close to 1 and $\pi<\theta<2\pi$, we see, as t *decreases* from θ to $-\theta$, that arg $p(re^{it})$ first *increases* by almost π as t passes through the value π, then *decreases* by almost π as t passes through 0, and finally *increases* by almost π again as t passes once more through π, going towards $-\theta$. The net change in arg $p(re^{it})$ is thus about π as t decreases from θ to $-\theta$, when $\pi<\theta<2\pi$. At the same time, we have $|p(re^{i\theta})|=|p(re^{-i\theta})|$.

For $|z|<1$, take

$$l(z)=[p(z)]^{(1/\pi i)\log((1-K)/(1+K))+1} \tag{2}$$

using the branch that makes $l(z)\to 1$ as $z\to i$, and for $\theta\neq 0, \pi$ we define $l(e^{i\theta})$ as $\lim_{r\to 1} l(re^{i\theta})$. The preceding discussion then shows that

$$l(e^{-i\theta})=[e^{i\pi}p(e^{i\theta})]^{(1/\pi i)\log((1-K)/(1+K))+1}=-\frac{1-K}{1+K}l(e^{i\theta})$$

from $\pi<\theta<2\pi$, whence

$$l(e^{i\theta})+\frac{1+K}{1-K}l(e^{-i\theta})=0,\qquad \pi<\theta<2\pi. \tag{3}$$

Now take any nonzero, continuously twice differentiable function $u(\theta)$, periodic of period 2π, and, in $[0, 2\pi]$, identically zero outside the interval $[\pi-a, \pi+a]$. We furthermore require, in $[\pi-a, \pi+a]$, that $u(\pi-t)=-u(\pi+t)$. It is then clear that $u(-\theta)=-u(\theta)$.

Let $U(z)$ be the function continuous in $|z|\leqq 1$ and harmonic in $|z|<1$ satisfying $U(e^{i\theta})=u(\theta)$, and let $V(z)$ be a harmonic conjugate of $U(z)$. Because $u(\theta)$ is so well behaved, $V(z)$ is actually continuous up to $|z|=1$ [**3**, Chapter VII], and is infinitely differentiable near 1 on $|z|=1$ because $\mathsf{U}(z)$ vanishes identically there. Add, if necessary, a constant to V so as to make $V(1)=0$; we will then have $V(z)=O(1-z)$ for $|z|\leqq 1$, $z\to 1$, and if $m(z)=\mathsf{U}(z)+iV(z)$, $m(z)$ is analytic in $|z|<1$, continuous up to $|z|=1$, and

$$m(z) = O(1-z) \qquad \text{for } z\to 1. \tag{4}$$

Since $\mathsf{U}(e^{i\theta})=u(\theta)$ is an *odd* function of θ, $V(e^{i\theta})$ is an *even* one [**3**, p. 50, formula(1)], and we see that

$$m(e^{-i\theta}) = m(e^{i\theta}) - 2u(\theta). \tag{5}$$

We take $F(z)=l(z)\ m(z)$. Formula (2) and the definition of $p(z)$ show that $l(z)$ is bounded in $|z|<1$, save near 1 where it is $O(1-z)^{-1}$. Therefore $F(z)$ is bounded in $|z|<1$ by (4); it is moreover clearly continuous up to $|z|=1$ save perhaps at 1 and -1.

For $\pi<\theta<2\pi$, we have from (3) and (5),

$$F(e^{i\theta}) + \frac{1+K}{1-K}F(e^{-i\theta}) = -2\frac{1+K}{1-K}l(e^{-i\theta})u(\theta). \tag{6}$$

By choice of u, the right side of (6) vanishes identically for $\pi+a<\theta<2\pi$. But it does not do so for $\pi<\theta<\pi+a$ because $u(\theta)$ does not vanish identically there (otherwise it would be identically zero), and $l(e^{-i\theta})$ does not, $l(z)$ being *analytic* and $\neq 0$ in $|z|<1$.

We have constructed a function F with the required properties, and the theorem is proved.

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Yeshiva College, New York, New York 10033