PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 29, No. 2, July 1971

# COUNTEREXAMPLE TO A QUESTION ON COMMUTATORS

ALAN McINTOSH

ABSTRACT. We show that it is possible for two selfadjoint operators $A$ and $B$ in a Hilbert space $H$ with bounded commutator $AB-BA$ to have the property that $|A|B-B|A|$ is unbounded (where $|A|$ denotes the positive square root of $A^2$). The proof reduces to showing that for all natural numbers $n$, there exist a bounded positive operator $U$ and a bounded operator $V$ satisfying $\|UV-VU\| \geqq n\|UV+VU\|$.

**Introduction.** Interest in the above question arises from the fact that if $H=L^2(-\infty, \infty)$, $Au=iu'$ and $Bu=bu$ (where $b$ is an a.e. differentiable function), then $|A|B-B|A|$ is bounded whenever $AB-BA$ is bounded (i.e. whenever $b'$ is essentially bounded). Note that $|A|B-B|A|$ is the singular integral operator $(|A|B-B|A|)f(x) = \pi^{-1}\,\text{p.v.} \int (x-y)^{-2}(b(x)-b(y))f(y)dy$. This is the one-dimensional $L^2$ case of a more general theorem of Calderon [1]. It was asked by T. Kato whether this case at least could be proved in an abstract setting, and in particular, whether $|A|B-B|A|$ is bounded whenever $AB-BA$ is bounded.

Although we present two operators with $AB-BA$ bounded and $|A|B-B|A|$ unbounded, the question remains as to whether an abstract proof of Calderon's result can be found. In particular it is clear that in the special case ($A=d/dx$, $B=b$), $AB-BA$ commutes with $B$. So it would be interesting to know the answer to the following question: If $AB-BA$ is bounded and commutes with $B$, is $|A|B-B|A|$ necessarily bounded? We comment further on this question at the end of the paper.

**Terminology.** If $A$ is a linear operator in a Hilbert space $H$, then $D(A)$ denotes the domain of $A$. A linear manifold $X \subset D(A)$ is called a *core* of $A$ if $X$ is dense in $D(A)$ under the norm $\|u\|_A^2 = \|u\|^2 + \|Au\|^2$. Throughout this paper the scalar field is assumed to be the field of complex numbers $\mathbf{C}$.

**The result.** (I) There exist two linear operators $A$ and $B$ in a Hilbert space $H$ satisfying:

(i) $A$ is selfadjoint;

Received by the editors October 15, 1970.

*AMS* 1970 *subject classifications*. Primary 47B47.

*Key words and phrases*. Hilbert space, selfadjoint operator, commutator.

(ii) $B$ is bounded and selfadjoint;
(iii) $B(D(A))\subset D(A)$, and $AB-BA$ is bounded on $D(A)$;
(iv) $|A|B-B|A|$ is unbounded.

[*Note.* (iii) is equivalent to (iii)′ $AB-BA$ is defined and bounded on a core of $A$ (provided that (i) and (ii) are satisfied).]

This result can be reduced to the following result on bounded operators:

(II) For all natural numbers $n$ there exist bounded selfadjoint operators $A_n$ and $B_n$ in a Hilbert space $H_n$ satisfying:
(i) $\|B_n\|\leqq 1$;
(ii) $\|A_nB_n-B_nA_n\|=1$;
(iii) $\|\,|A_n|B_n-B_n|A_n|\,\|\geqq n$.

Indeed (I) is derived from (II) by setting $H=\bigoplus H_n$, $A=\bigoplus A_n$ and $B=\bigoplus B_n$. (Note that the set $\{u=\bigoplus u_n|$ all but finitely many of the $u_n$ are zero$\}$ is a core of $A$.)

Result (II) is a consequence of (III):

(III) For all natural numbers $n$, there exist bounded linear operators $U$ and $V$ in a Hilbert space $K$ satisfying:
(i) $U$ is positive selfadjoint;
(ii) $\|V\|\leqq 1$;
(iii) $\|UV+VU\|=1$;
(iv) $\|UV-VU\|\geqq n$.

To derive (II) from (III), let $H_n=K\oplus K$ (with elements of $H_n$ denoted by $u_1\oplus u_2$), let $A_n=U\oplus(-U)$, and define $B_n$ by $B_n(u_1\oplus u_2) = Vu_2\oplus V^*u_1$ We can represent $A_n$ and $B_n$ by the matrices

$$A_n=\begin{bmatrix} U & 0\\ 0 & -U\end{bmatrix} \quad\text{and}\quad B_n=\begin{bmatrix} 0 & V\\ V^* & 0\end{bmatrix}.$$

Then

$$A_nB_n-B_nA_n=\begin{bmatrix} 0 & UV+VU\\ -(UV^*+V^*U) & 0\end{bmatrix},$$

$$|A_n|=\begin{bmatrix} U & 0\\ 0 & U\end{bmatrix} \quad\text{and}$$

$$|A_n|B_n-B_n|A_n|=\begin{bmatrix} 0 & UV-VU\\ UV^*-V^*U & 0\end{bmatrix}.$$

It is now straightforward to check that (II) is a consequence of (III).

It remains for us to prove (III). Let $k=(3n+1)^2$ and let $m=2^k$. Let $K$ be the $m$-dimensional unitary space $\mathbf{C}^m$. Define $U$ to be the operator in $K$ whose matrix is diagonal with diagonal terms $\lambda_i=2^i$ $(i=1,2,\cdots,m)$. The operator $V$ is defined to have matrix $(v_{ij})$, where $v_{ij}=(\lambda_i+\lambda_j)^{-1}w_{ij}$, and where the matrix $W=(w_{ij})$ is defined inductively by

$$W_1=\begin{bmatrix}\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}}\\ 0 & 0\end{bmatrix},\qquad W_{r+1}=\begin{bmatrix}W_r+Y_r & -Y_r\\ 0 & W_r\end{bmatrix},$$

and $W=W_k$. The matrix $Y_r$ is the $2^r\times 2^r$ matrix with terms in the bottom row equal to $2^{-(r+1)/2}$, and all other terms zero. So, for example,

$$W_2=\begin{bmatrix}\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & 0\\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2}\\ 0 & 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}}\\ 0 & 0 & 0 & 0\end{bmatrix}.$$

The matrix $W$ is constructed so that (i) the nonzero rows, considered as vectors in $K$, form an orthonormal set, (ii) if $W_a$ is the matrix $(|w_{ij}|)$, then $\|W_a(1,1,1,\cdots,1)\|=\sqrt{(km)}=\sqrt{k}\|(1,1,1,\cdots,1)\|$, and (iii) $w_{ij}\geqq 0$ if $i\geqq j$, and $\leqq 0$ if $i<j$.

We now check the four properties listed in (III). The first is clear.

(ii)
$$\begin{aligned}\|V\|^2 &\leqq \sum_i\sum_j v_{ij}^2=\sum_i\sum_j(\lambda_i+\lambda_j)^{-2}w_{ij}^2\\ &\leqq \sum_i \lambda_i^{-2}\sum_j w_{ij}^2<1.\end{aligned}$$

(iii) The fact that the nonzero rows of $W$ form an orthonormal set implies that $\|W\|=1$. Now $W=UV+VU$, so $\|UV+VU\|=1$.

(iv) Denote $UV-VU$ by $S=(s_{ij})$. Then

$$s_{ij}=(\lambda_i-\lambda_j)v_{ij}=(\lambda_i-\lambda_j)(\lambda_i+\lambda_j)^{-1}w_{ij}\geqq \tfrac{1}{3}|w_{ij}|\quad \text{if } i\neq j$$

(because $(\lambda_i-\lambda_j)(\lambda_i+\lambda_j)^{-1}\geqq\frac{1}{3}$ if $i>j$, and $\leqq -\frac{1}{3}$ if $i<j$).

$$\therefore\ s_{ij}+\tfrac{1}{3}\delta_{ij}\geqq\tfrac{1}{3}|w_{ij}|,\qquad 1\leqq i,j\leqq m.$$

$$\therefore\ \|(S+\tfrac{1}{3}I)(1,1,\cdots,1)\|\geqq\tfrac{1}{3}\|W_a(1,1,\cdots,1)\|=\tfrac{1}{3}\sqrt{(km)}.$$

$$\therefore\ \|S(1,1,\cdots,1)\|\geqq\tfrac{1}{3}(\sqrt{k}-1)\sqrt{m}.$$

$$\therefore\ \|S\|\geqq n\qquad(\because\ k=(3n+1)^2).$$

This completes the proof.

**Remark.** It was asked in the introduction whether the situation is altered by adding the extra condition: $B$ commutes with $AB-BA$.

It is interesting to note that a counterexample cannot be constructed along the above lines in this case. For if $A$ and $B$ are *bounded* selfadjoint operators such that $B$ commutes with $AB-BA$, then $AB-BA=0$ (see [**2**, p. 4]).

ADDED IN PROOF. 1. We may require that the $V$ defined in (III) be selfadjoint (for if $V$ satisfies the equations in (III), then either the real or imaginary part of $V$ satisfies the same equations with $n$ replaced by $\frac{1}{2}n$). Hence there exist selfadjoint operators $U$ and $V$ in $\mathbf{C}^m$ such that $U$ is positive and $\|UV-VU\|/\|UV+VU\| \geqq \text{const}(\log m)^{1/2}$. The author has learned by private communication that W. Kahan has obtained such operators satisfying $\|UV-VU\|/\|UV+VU\| \geqq \pi^{-1} \log m$. Kahan has also shown that if $Z$ is an $m \times m$ matrix with real eigenvalues, then $\|Z-Z^*\|/\|Z+Z^*\| \leqq 0.1+\log_2 m$.

2. The author has applied (III) to prove the existence of a regularly accretive operator $A$ such that $A^{1/2}$ and $A^{*(1/2)}$ have different domains. Details will be published elsewhere.

## REFERENCES

1. A. P. Calderon, *Commutators of singular integral operators*, Proc. Nat. Acad. Sci. U.S.A. **53** (1965), 1092–1099. MR **31** #1575.

2. C. R. Putnam, *Commutation properties of Hilbert space operators and related topics*, Springer-Verlag, New York, 1967. MR **36** #707.

MACQUARIE UNIVERSITY, NORTH RYDE, N.S.W., 2113, AUSTRALIA