

ON A SEQUENCE OF ALMOST DETERMINISTIC PAIRWISE INDEPENDENT RANDOM VARIABLES

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ABSTRACT. We show the existence of infinite sequences of pairwise independent random variables (which are not constant almost everywhere) such that the whole sequence is determined with probability one by the knowledge of any pair of members of that sequence.

1. Introduction. We want to show the existence of infinite sequences $Y_1, Y_2, \dots, Y_n, \dots$ of pairwise independent random variables (which are not constant almost everywhere) having the property that for any $i \neq j$ there exist Borel functions g_n such that

$$Y_n = g_{i,j,n}(Y_i, Y_j), \quad n = 1, 2, 3, \dots$$

This answers a question raised by D. Dawson. For finite sets, such a construction is done in [3].

2. Construction. We will first construct a sequence of random elements having the properties mentioned in the introduction. Let G be the character group of Q , the additive group of the rational numbers with the discrete topology. G is an abelian compact group which is divisible and torsion free. Moreover G is metrizable, complete and separable. There is a lot of information in the literature about G , but we refer only to [2] where one can find other references. G being compact, the Haar measure μ is uniquely determined on the Borel sets of G by $\mu(G) = 1$. Take as probability space $\Omega = G \times G$ the direct product of the group G with itself, and for measure on Ω take the Haar measure $\mu \times \mu$ which is the product measure of μ with itself. Let X_1 (resp. X_2) denote the projection maps of $G \times G$ onto the first (resp. second) coordinates. X_1 and X_2 are random independent elements taking values in G . Define

$$(1) \quad X_n = (n - 2)X_1 + X_2, \quad n = 2, 3, \dots$$

Given any $i \neq j$ it is easily verified, since G is a vector space over the rationals, that the relations given by (1) define X_1 and X_2 uniquely in terms of X_i and X_j ; moreover if we write $(X_1, X_2) = g_{i,j}(X_i, X_j)$, $g_{i,j}$

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is an automorphism of Ω . Let us compute the image $\tilde{\mu}$ of μ under this automorphism. For any set $A \times B$ of Ω where A and B are Borel sets of G we have

$$\begin{aligned}\tilde{\mu}(A \times B) &= \mu \times \mu \{(X_i, X_j) \in A \times B\} \\ &= \mu \times \mu \{(X_i, X_j) \in g_{i,j}^{-1}(A \times B)\}.\end{aligned}$$

Since $g_{i,j}$ is a group isomorphism it follows that $\tilde{\mu}(A \times B)$ is translation-invariant and therefore $\tilde{\mu}$ is a Haar measure; since g is onto $\tilde{\mu}(G \times G) = 1$ and $\tilde{\mu} = \mu \times \mu$. It follows that

$$\mu \times \mu \{(X_i, X_j) \in A \times B\} = \mu(X_i \in A) \cdot \mu(X_j \in B) = \mu(A) \cdot \mu(B).$$

Our sequence $X_1, X_2, \dots, X_n, \dots$ is a sequence of random elements having the properties stated in the introduction. To establish the existence of random variables having these properties, it suffices to note that G being complete and separable there is a Borel isomorphism ϕ of G on $[0, 1]$ where the image of μ is Lebesgue measure. See for instance [1, p. 339]. If we define $Y_n = \phi(X_n)$ we have established the following

THEOREM. *The sequence Y_1, \dots, Y_n, \dots defines pairwise independent random variables. Each of the Y_i is uniformly distributed on $[0, 1]$ and for each $i, j, i \neq j$, there are Borel functions $g_{i,j,n}$ such that $Y_n = g_{i,j,n}(Y_i, Y_j), n = 1, 2, \dots$*

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