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SOLVING INTEGRAL EQUATIONS BY L AND L^{-1} OPERATORS

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ABSTRACT.

$$g(u) = \int_0^\infty k(ux)f(x)\,dx = \frac{1}{2\pi i}\int_C K(s)F(1-s)u^{-s}ds, \tag{1}$$

where $g(u)$ and $k(u)$ are known and $f(x)$ is to be found. $K(s)$ is the Mellin transform of $k(x)$ and $F(s)$ of $f(x)$; hence the second equality. L and L^{-1} denote the Laplace transform and its inverse. If

$$K(s) = \prod_{i=1}^{n} \Gamma(\alpha_i s + \beta_i) \Big/ \prod_{j=1}^{m} \Gamma(\alpha_j s + \beta_j) \tag{2}$$

then I show that a suitable combination of L and L^{-1} operators, applied to (1), can eliminate $K(s)$ from the second integrand. This leaves $F(1-s)$ standing free and the Mellin transform then obtains $f(x)$ from $F(1-s)$. This solution needs tables of Laplace transforms only.

When (2) does not hold, an L and L^{-1} combination may turn (1) into an integral equation whose solution is already known.

1. Introduction.

L is the Laplace transform defined by

$$L\{\phi(x)\} = \int_0^\infty e^{-xt}\phi(x)\,dx = \psi(t). \tag{1}$$

L^{-1} is the inverse of L. With $\phi(x)$ and $\psi(t)$ as in (1) we then have

$$L^{-1}\{\psi(t)\} = \phi(x). \tag{2}$$

Our aim is to show that a large variety of integral equations can be solved by means of the operators L and L^{-1}.

Given $\psi(t)$, we can evaluate $L^{-1}\{\psi(t)\}$ by several methods. One is by complex integration [**8**, p. 66, Theorem 7.3]; another is by using Post's operator [**8**, p. 277, (6)] which uses real variable methods only.

Here we assume that $L^{-1}\{\psi(t)\}$ is found by reading a table of Laplace transforms in reverse. We then have

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$$L^{-1}[L\{\phi(x)\}] = \phi(x). \tag{3}$$

2. **The annihilating power of L^{-1}.** Our basic result is as follows:

THEOREM. *If*

(i) $\alpha > 0$, $\frac{1}{2}\alpha + \beta > 0$, $t > 0$;

(ii) $s = \sigma + i\mu$, *σ and μ both real*; $F(s) \in L(\frac{1}{2} - i\infty, \frac{1}{2} + i\infty)$

then

$$L^{-1}\left\{\frac{1}{2\pi i}\int_C \Gamma(\alpha s + \beta)F(s)t^{-\alpha s-\beta}ds\right\} = \frac{1}{2\pi i}\int_C F(s)x^{\alpha s+\beta-1}ds, \tag{4}$$

where, for both integrals, the contour C may be the line $\sigma = \frac{1}{2}$, a line parallel to the imaginary axis in the complex s plane.

PROOF. Let R denote the right-hand side of (4). When C is the line $\sigma = \frac{1}{2}$ we see from $\alpha > 0$ and $F(s) \in L(\frac{1}{2} - i\infty, \frac{1}{2} + i\infty)$ that the s-integral in R, with the factor $x^{\beta-1}$ excluded, is absolutely convergent for all values of x.

Let C be the line $\sigma = \frac{1}{2}$ and consider

$$L\{R\} = \int_0^\infty e^{-xt}\left\{\frac{1}{2\pi i}\int_C F(s)x^{\alpha s+\beta-1}ds\right\}dx. \tag{5}$$

The modulus of the integrand is $e^{-xt}|F(s)|x^{\alpha/2+\beta-1}$. Hence, from the conditions, the right-hand side of (5) is an absolutely convergent double integral. Consequently we may change the order of integration and integrate first with respect to x. This gives us

$$L\{R\} = \frac{1}{2\pi i}\int_C \Gamma(\alpha s + \beta)F(s)t^{-\alpha s-\beta}ds. \tag{6}$$

From equation (3), and the remark prior to it, an application of L^{-1} to both sides of (6) gives us

$$R = L^{-1}\left\{\frac{1}{2\pi i}\int_C \Gamma(\alpha s + \beta)F(s)t^{-\alpha s-\beta}ds\right\}, \tag{7}$$

where C can be the line $\sigma = \frac{1}{2}$. This completes the proof of (4).

3. **Some corollaries of** (4). s-integrals and integrands, as in (4), will be called Mellin type integrals and Mellin type integrands.

(i) (4) shows that when L^{-1} acts on a Mellin type integral it can eliminate the factor $\Gamma(\alpha s + \beta)$ from the numerator of the integrand.

(ii) In (4) write $F(s) = H(s)/\Gamma(\alpha s + \beta)$ and cancel out the common factor $\Gamma(\alpha s + \beta)$ in the left-hand side. We then see that when L^{-1}

acts on a Mellin type integral it can introduce a new factor $\Gamma(\alpha s+\beta)$ in the denominator of the integrand.

(iii) R denotes the right-hand side of (4). Hence (6) shows that when L acts on a Mellin type integral it can introduce a new factor $\Gamma(\alpha s+\beta)$ into the numerator of the integrand.

(iv) Write $F(s)=H(s)/\Gamma(\alpha s+\beta)$ in (6) and cancel out the common factor $\Gamma(\alpha s+\beta)$ on the right-hand side. Then we can see that L acting on a Mellin type integral can eliminate $\Gamma(\alpha s+\beta)$ from the denominator of the integrand.

In the special case when $\alpha=1$ and $\beta=0$, (iv) is given in [**1**, p. 132, transform (29)].

The eliminating powers of L and L^{-1} in (i) and (iv) enable us to solve a wide variety of integral equations.

4. **The Mellin transform.** In some of our applications we shall need the Mellin transform and some related theory. If

$$F(s) = \int_0^\infty f(x)x^{s-1}dx \tag{8}$$

then we say that $F(s)$ is the Mellin transform of $f(x)$.

If (8) holds, then the inverse Mellin transform is given by

$$f(x) = \frac{1}{2\pi i}\int_C F(s)x^{-s}ds, \tag{9}$$

where s is a complex variable and C is some suitable contour.

If $F(s)$ and $G(s)$ are, respectively, the Mellin transforms of $f(x)$ and $g(x)$ then we also have

$$\int_0^\infty f(x)g(x)\,dx = \frac{1}{2\pi i}\int_C F(s)G(1-s)\,ds, \tag{10}$$

sometimes known as the Parseval theorem for Mellin transforms.

Conditions and proofs for (8), (9), and (10) can be found in [**4**, p. 46, §1.29, p. 60, §2.7 and p. 94, §3.17].

5. **Solving integral equations by means of the L and L^{-1} operators.** Consider the integral equation

$$g(u) = \left(\frac{2}{\pi}\right)^{1/2}\int_0^\infty \sin(ux)f(x)\,dx, \tag{11}$$

where $g(u)$ is known and $f(x)$ is to be found.

For this example we must first express the integral in (11) in the form of a Mellin type integral and this can be done by using the

Parseval theorem (10). The application of (10) is justified by [4, p. 60, Theorem 43] with $k=\frac{1}{2}$, if $f(x)$ and $f(x)x^{-1/2}$ both belong to $L(0, \infty)$ and $F(s)$, the Mellin transform of $f(x)$, belongs to $L(\frac{1}{2}-i\infty, \frac{1}{2}+i\infty)$. The last condition will also allow us to use the theorem in §2.

The Mellin transform of sin x is given by [4, p. 196, (7.9.3)] with $\nu=\frac{1}{2}$, and for $\sin(ux)$ we must multiply by u^{-s}. Hence, the application of (10) to (11) gives us

$$g(u) = \frac{1}{2\pi i}\int_C 2^{s-1/2} \frac{\Gamma(\frac{1}{2}+\frac{1}{2}s)}{\Gamma(1-\frac{1}{2}s)} F(1-s)u^{-s}ds, \tag{12}$$

where, since we can take $k=\frac{1}{2}$ in Theorem 43, we may also take C to be the line $\sigma=\frac{1}{2}$.

Consider now $L\{g(u^{1/2})\}$, which is a double integral in u and s. If $F(s)$ belongs to $L(\frac{1}{2}-i\infty, \frac{1}{2}+i\infty)$ so does $F(1-s)$. Hence, from the asymptotic expansion of the Gamma function [7, p. 273] and with the s-integral taken along $\sigma=\frac{1}{2}$, this double integral is absolutely convergent. On integrating first with respect to u the factor $\Gamma(1-\frac{1}{2}s)$ is cancelled out and we obtain

$$L\{g(u^{1/2})\} = \frac{1}{2\pi i}\int_C 2^{s-1/2}\Gamma(\tfrac{1}{2}+\tfrac{1}{2}s)F(1-s)t^{-1+s/2}ds. \tag{13}$$

In order to use (4) we write $t=1/\tau$ in (13). We then have

$$\begin{aligned}&\tau^{-3/2}\{[L\{g(u^{1/2})\}]_{t=1/\tau}\}\\ &\qquad = \frac{1}{2\pi i}\int_C 2^{s-1/2}\Gamma(\tfrac{1}{2}+\tfrac{1}{2}s)F(1-s)\tau^{-1/2-s/2}ds.\end{aligned} \tag{14}$$

The conditions of the theorem in §2 are satisfied, with $\alpha=\beta=\frac{1}{2}$. Hence, from (4), we have

$$L^{-1}[\tau^{-3/2}\{[L\{g(u^{1/2})\}]_{t=1/\tau}\}] = \frac{1}{2\pi i}\int_C 2^{s-1/2}F(1-s)x^{s/2-1/2}ds, \tag{15}$$

$$= \frac{2^{1/2}}{2\pi i}\int_C F(s)(2x^{1/2})^{-s}ds, \tag{16}$$

on replacing s by $1-s$ in (15). If, in (15), C is the line $\sigma=\frac{1}{2}$ then this replacement leaves the contour of integration unaltered.

Finally, on applying (9) to the right-hand side of (16), our solution of (11) becomes

$$L^{-1}[\tau^{-3/2}\{[L\{g(u^{1/2})\}]_{t=1/\tau}\}] = 2^{1/2}f(2x^{1/2}). \tag{17}$$

To illustrate (17) let $g(u)=0$ when $0<u<a$ and $g(u)=(u^2-a^2)^{-1/2}$ when $u>a$. $L\{g(u^{1/2})\}=\pi^{1/2}t^{-1/2}e^{-a^2t}$, using [1, p. 137, (5)]. From [1, p. 245, (40)] and (17) we then have

$$2^{1/2}f(2x^{1/2}) = L^{-1}[\pi^{1/2}\tau^{-1}e^{-a^2/\tau}], \tag{18}$$

$$= \pi^{1/2}J_0(2ax^{1/2}). \tag{19}$$

The solution of (11) with the above choice of $g(u)$ is then

$$f(x) = (\pi/2)^{1/2}J_0(ax). \tag{20}$$

This can be verified from the Fourier sine transforms given in [1, p. 99, (1)].

The classical solution of (11) is

$$f(x) = \left(\frac{2}{\pi}\right)^{1/2}\int_0^\infty \sin(xu)g(u)\,du. \tag{21}$$

For our choice of $g(u)$ the solution of (11) can be obtained from (21) only by evaluating a somewhat difficult integral.

If $J_\nu(x)$ is the Bessel function of order ν and $\nu>-1$ then the general Hankel transform is

$$g(u) = \int_0^\infty (xu)^{1/2}J_\nu(xu)f(x)\,dx. \tag{22}$$

This reduces to (11) when $\nu=\frac{1}{2}$. Considered as an integral equation for $f(x)$, the classical solution of (22) is obtained by interchanging $f(x)$ and $g(u)$, in (22), and replacing the variable x of integration by u. This solution bears the same relation to (22) as (21) does to (11).

The L and L^{-1} solution of (22), analogous to (17), is

$$2^{1/2}x^{\nu/2-1/4}f(2x^{1/2}) = L^{-1}[\tau^{-\nu-1}\{[L\{u^{\nu/2-1/4}g(u^{1/2})\}]_{t=1/\tau}\}]. \tag{23}$$

On applying the operator L to both sides of (23) we obtain a result equivalent to one discovered by Tricomi [5, equations (9), (9′) and (15)], all of which are equivalent to each other.

6. Some formal L and L^{-1} solutions of other types of integral equations. $K_\nu(x)$ denotes the associated Bessel function, see [6, p. 78, (6)], and the Mellin transform of $x^\nu K_\nu(x)$ is $2^{s+\nu-2}\Gamma(\frac{1}{2}s)\Gamma(\frac{1}{2}s+\nu)$, [4, p. 197, (7.9.12)]. Let

$$g(t) = \int_0^\infty (ut)^\nu K_\nu(ut)f(u)\,du, \tag{24}$$

where $g(t)$ is given and $f(u)$ is to be found. Proceeding formally we apply the Parseval theorem (10) to the right-hand side of (24) and obtain

$$(25) \qquad g(t) = \frac{1}{2\pi i}\int_C 2^{s+\nu-2}\Gamma(\tfrac{1}{2}s)\Gamma(\tfrac{1}{2}s+\nu)F(1-s)t^{-s}ds,$$

where $F(s)$ is the Mellin transform of $f(x)$. If we may use (4) then $L^{-1}\{g(t^{1/2})\}$ eliminates $\Gamma(\frac{1}{2}s)$ from (25) and then, after writing $x=1/t$, $L^{-1}t^{-\nu-1}$ eliminates $\Gamma(\frac{1}{2}s+\nu)$. The final result is

$$(26) \quad L^{-1}[t^{-\nu-1}\{[L^{-1}\{g(t^{1/2})\}]_{x=1/t}\}] = \frac{2^{\nu-1}x^{\nu-1/2}}{2\pi i}\int_C F(1-s)(2x^{1/2})^{s-1}ds,$$

$$(27) \qquad = 2^{\nu-1}x^{\nu-1/2}f(2x^{1/2}),$$

where, in (26), s is replaced by $1-s$ and then (9) is used to obtain (27), as in going from (16) to (17). (27) is then our solution of (24).

If, e.g., $g(t)=1/(t^2+a^2)$ then, from [1, p. 229, (1)], we have $L^{-1}\{g(t^{1/2})\}=\exp(-a^2x)$ and from [1, p. 245, (40)],

$$(28) \qquad L^{-1}\{t^{-\nu-1}e^{-a^2/t}\} = a^{-\nu}x^{\nu/2}J_\nu(2ax^{1/2}).$$

From (27), for this choice of $g(t)$, our solution of (24) is

$$(29) \qquad f(x) = a^{-\nu}x^{1-\nu}J_\nu(ax).$$

This solution is easily checked from the K_ν tables [2, p. 137, (16)] with $\nu=-\mu$, noting that the tables have $(ut)^{1/2}$ in the integrand of (24) instead of $(ut)^\nu$ as we have.

7. **Some general remarks.** Suppose that we have the integral equation

$$(30) \qquad g(u) = \int_0^\infty k(ux)f(x)\,dx,$$

where $g(u)$ and $k(ux)$ are given and $f(x)$ is to be found. On applying the Parseval theorem for Mellin transforms (10) to the right-hand side of (30) we find that instead of having one Gamma function in the numerator and one in the denominator, as in (12), we have $\prod_{i=1}^n \Gamma(\alpha_i s+\beta_i)$ in the numerator and $\prod_{j=1}^m \Gamma(\alpha_j s+\beta_j)$ in the denominator. Then we can proceed as follows:

(i) by means of an m succession of L operators we can eliminate all the Gamma functions from the denominator, as in the elimination from (12) to (13) and

(ii) by means of an n succession of L^{-1} operators we can eliminate all the Gamma functions from the numerator, as in going from (25) to (26).

If convergence conditions allow all these operations to be performed we shall then have $F(1-s)$ standing free, as in (26). Since $F(s)$ is the Mellin transform of $f(x)$ we can then express the final integral obtained in terms of $f(x)$, as in going from (26) to (27), and so obtain the solution of (30). This method enables us to solve a large variety of integral equations by using tables of Laplace transforms only.

In the next section we give another useful method of solving integral equations by using the L and L^{-1} operators.

8. **The reduction of an integral equation to one whose solution is known.** We illustrate the method by means of an example. Consider the Stieltjes integral equation

$$g(t) = \int_0^\infty (u+t)^{-\alpha} f(u)\,du, \qquad \alpha > 0, \tag{31}$$

where $g(t)$ is given and $f(u)$ is to be found. By applying the operator L to both sides and using (1), it is easy to establish that $L^{-1}\{(u+t)^{-\alpha}\} = x^{\alpha-1}e^{-ux}/\Gamma(\alpha)$. Hence, if we may interchange the order of applying L^{-1} to the right-hand side of (31) and infinite integration, we have

$$L^{-1}\{g(t)\} = \frac{x^{\alpha-1}}{\Gamma(\alpha)} \int_0^\infty e^{-ux} f(u)\,du. \tag{32}$$

(31) is then reduced to an integral equation which can be solved by standard methods, since (32) can be solved by an application of L^{-1}. We now apply L^{-1} to (32), using (2), and obtain

$$L^{-1}\{\Gamma(\alpha)t^{1-\alpha}([L^{-1}\{g(t)\}]_{x=t})\} = f(x) \tag{33}$$

as our solution of (31).

In order to test (33) take $\alpha = 1$ and

$$g(t) = a^{-\nu/2}t^{\nu/2}K_\nu(2a^{1/2}t^{1/2}), \qquad a > 0, \tag{34}$$

where $K_\nu(x)$ is the associated Bessel function, as in §6 and, for convergence $-1<\nu<3/2$. Then the first L^{-1} of (33) is given by the inverse Laplace tables [**1**, p. 283, (40)], and the second L^{-1} from [**1**, p. 245, (40)]. With $g(t)$ as in (34) our solution of (31) is then

$$f(x) = 2^{-1}a^{-\nu/2}x^{\nu/2}J_\nu(2a^{1/2}x^{1/2}) \tag{35}$$

where $a>0$ and $-1<\nu<3/2$ and $J_\nu(x)$ is a Bessel function of order ν. This is easily checked from the Stieltjes transform tables [2, p. 225, (10)], with $k=0$.

Studies of (31), from a point of view quite different from that given here, are found in Widder [**8**, Chapter VIII], Erdélyi [**3**], and in [**2**, p. 237, (33)] a general solution of (31) is given with $g(t)$ and $f(u)$ both expressed as hypergeometric functions.

The reduction of (31) to an integral equation whose solution can be obtained by known methods requires an application of L^{-1}. A general application of this method will usually require a combination of L and L^{-1} operators to make a successful reduction of this nature.

9. **Some further general remarks.** In §3 we have shown that L and L^{-1}, when acting upon Mellin type integrals, have the power to annihilate or to introduce Gamma function factors into the integrand. Here we have only used the annihilating powers of L and L^{-1}. But the power of introducing new Gamma function factors is also useful and leads to many interesting results. For example this power can be used to show that (17) and (21), the two solutions of (11), although apparently quite different from each other are in fact identical.

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