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TOPOLOGICAL SPACES WITH A σ-POINT FINITE BASE

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ABSTRACT. The principal results of the paper are as follows. A topological space with a σ-point finite base has a σ-disjoint base if it is either hereditarily collectionwise normal or hereditarily screenable. From a metrization theorem of Arhangel'skiĭ, it follows that a T_1-space with a σ-point finite base is metrizable iff it is perfectly normal and collectionwise normal. A topological space with a σ-point base is quasi-developable in the sense of Bennett. Consequently a theorem of Čoban follows that for a topological space $(X, \mathfrak{J})$ the following are equivalent: (a) $(X, \mathfrak{J})$ is a metacompact normal Moore space. (b) $(X, \mathfrak{J})$ is a perfectly normal T_1-space with a σ-point finite base.

1. **Introduction.** Recently there has been a renewed interest in topological spaces with a point-countable base. See for instance Aleksandrov [**1**], Corson and Michael [**8**], and Heath [**9**]. In this paper we propose to study a subfamily of these spaces, the family of topological spaces with a σ-point finite base. Sion and Zelmer [**16**] and Norman [**15**] proved that a T_3-space with a σ-point finite base is quasi-metrizable, and Arhangel'skiĭ [**2**] proved that every perfectly normal, collectionwise normal T_1-space with a σ-point finite base is metrizable. Here it is proved that spaces with a σ-point finite base are quasi-developable in the sense of Bennett [**4**]. A corollary is that a topological space is a normal metacompact Moore space iff it is a perfectly normal T_1-space with a σ-point finite base. This result also follows from theorems of Čoban [**7**, Theorem **11**] and Burke [**6**, Theorem 1.2]. We will show, too, that a hereditarily screenable or a hereditarily collectionwise normal space with a σ-point finite base has a σ-disjoint base.

2. **σ-discrete refinements.**

LEMMA 1. *Let $\mathcal{V}$ be an open point-finite family of subsets of a topological space $(X, \mathfrak{J})$. Then there exists a σ-disjoint family $\mathfrak{M} = \bigcup_1^\infty \mathfrak{M}_n$ where each $\mathfrak{M}_n$ is discrete with respect to some open subspace G_n. Further-*

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more if $x \in V \in \mathcal{V}$, *there exists* $M \in \mathfrak{M}$ *such that* $x \in M \subset V$ *and such that only finitely many members* V *of* $\mathcal{V}$ *contain* M.

PROOF. Let $\mathcal{V}_x = \{V: x \in V\}$. The relation $x \sim y$ iff $\mathcal{V}_x = \mathcal{V}_y$ is an equivalence relation on $W = \bigcup \{V: V \in \mathcal{V}\}$. Set $[x] = \{y: y \sim x\}$. ($[x]$ consists of all elements of X in exactly the same members of $\mathcal{V}$ as x.) Let X_n consist of all elements of X in exactly n members of $\mathcal{V}$. Set $\mathfrak{M}_n = \{[x]: x \in X_n\}$; $G_x = \bigcap \{V: [x] \subset V \in \mathcal{V}\}$ and $G_n = \bigcup \{G_x: x \in X_n\}$. The family $\mathfrak{M}_n$ is discrete with respect to G_n.

THEOREM 1. *Let* $(X, \mathfrak{J})$ *be perfectly normal and metacompact. Then every open cover of* X *has a* σ*-discrete closed refinement, i.e.* $(X, \mathfrak{J})$ *is subparacompact* [**6**].

This theorem follows from the preceding lemma.

Since Čoban [**7**] has proved that metacompact spaces such that every closed set is a G_δ are σ-paracompact in the sense of Arhangel'skiĭ and Burke has shown that σ-paracompact spaces are subparacompact, we omit the proof. In fact "subparacompact" is synonymous with F_σ-screenable and σ-paracompact, which Burke has shown to be equivalent [**6**, p. 655].

Burke [**6**] has an example of a locally compact metacompact T_2-space such that no open cover has a σ-discrete refinement. In view of Theorem 1 it would be interesting if there was a T_4-space with the above properties.

Conversely, there exists a subparacompact space that is not metacompact. See Michael [**12**, p. 278] in regard to Example H of Bing [**5**]. This example is T_4 and the countable union of closed paracompact subspaces, and hence is subparacompact by a theorem of Burke [**6**]. Michael has shown this example is not metacompact.

THEOREM 2. *Let* $(X, \mathfrak{J})$ *be a hereditarily collectionwise normal space with a* σ*-point finite base. Then* $(X, \mathfrak{J})$ *has a* σ*-disjoint base.*

PROOF. Let $\mathcal{V}$ be a point finite family of the base. We apply Lemma 1. Since $(X, \mathfrak{J})$ is hereditarily collectionwise normal, there is a family of pairwise disjoint open sets $\mathcal{W}_n$ such that if $M \in \mathfrak{M}_n$ there exists $W \in \mathcal{W}_n$ such that $M \subset W$. Let $\mathcal{S}_n = \{\bigcap \{W \cap V: V \in \mathcal{V}, M \subset V, M \subset W\}: M \in \mathfrak{M}_n\}$ and let $\mathcal{S} = \bigcup_{n=1}^{\infty} \mathcal{S}_n$. If $x \in V \in \mathcal{V}$, there exists $M \in \mathfrak{M}$ such that $x \in M \subset V$. Hence by the construction of $\mathcal{S}_n$ there exists $S \in \mathcal{S}$ such that $x \in S \subset V$. As $\mathcal{S}_n$ is a pairwise disjoint family of open sets, the theorem follows.

COROLLARY 2A (ARHANGEL'SKIĬ). *A perfectly normal, collectionwise normal* T_1*-space with a* σ*-point finite base is metrizable.*

PROOF. It can be easily shown that a perfectly normal space with a σ-disjoint base has a σ-discrete base. For instance, see Aull [3].

Čoban [7] has given another proof of the above result of Arhangel'skiĭ.

COROLLARY 2B. *A hereditarily countably paracompact space* $(X, \mathfrak{J})$ *with a σ-point finite base has a σ-disjoint base iff* $(X, \mathfrak{J})$ *is hereditarily paracompact.*

PROOF. Nagami [14] has proved that countably paracompact screenable spaces are paracompact.

3. Quasi-developable spaces.

DEFINITION 1 [4]. A sequence $\mathcal{G}_1, \mathcal{G}_2, \cdots$ of collections of open subsets of a topological $(X, \mathfrak{J})$ is called a quasi-development for $(X, \mathfrak{J})$ provided that if $x \in T \in \mathfrak{J}$, there exists n and G such that $x \in G \in \mathcal{G}_n$, and if $x \in G \in \mathcal{G}_n$ then $G \subset T$. We will refer to each $\mathcal{G}_k$ as a collection of the quasi-development.

THEOREM 3. *Let* $(X, \mathfrak{J})$ *have a σ-point finite base. Then* $(X, \mathfrak{J})$ *has a quasi-development.*

PROOF. Let $\mathcal{V}_k$ be a point-finite family. Let $X_{k,n}$ consist of all $x \in X$ in exactly n numbers of $\mathcal{V}_k$. Let $W_{k,x} = \bigcap \{ V : x \in V \in \mathcal{V}_k \}$. Let $\mathcal{W}_{k,n} = \{ W_{k,x} : x \in X_{k,n} \}$. The family $\mathcal{W} = \bigcup_k \bigcup_n \mathcal{W}_{k,n}$ is a quasi-development for $(X, \mathfrak{J})$.

Bennett [4] has shown that hereditarily metacompact and quasi-developable spaces have a point-countable base. We prove a stronger result.

THEOREM 4. *Let* $(X, \mathfrak{J})$ *be hereditarily metacompact (hereditarily screenable) and quasi-developable. Then* $(X, \mathfrak{J})$ *has a σ-point finite base (σ-disjoint base). In fact it has a quasi-development where each collection of the quasi-development is point finite (pairwise disjoint).*

PROOF. We prove only the part involving the hereditarily metacompactness. For each collection $\mathcal{G}_n$ of the quasi-development, let $G_n = \bigcup \{ G : G \in \mathcal{G}_n \}$. Let $\mathcal{W}_n$ be a point-finite open refinement of $\mathcal{G}_n$ that covers G_n. Since G_n is open the cover and refinement can be considered as either relatively open with respect to G_n or $\mathfrak{J}$-open. If $x \in T \in \mathfrak{J}$, there exists n and G such that $x \in G \in \mathcal{G}_n$, and if $x \in G \in \mathcal{G}_n$ then $G \subset T$. So if $x \in W \in \mathcal{W}_n$, $W \subset T$, since $W \subset G$ for some G such that $G \subset T$.

From this theorem an important result of Čoban follows.

THEOREM 5 (ČOBAN). *For a topological space* $(X, \mathfrak{I})$ *the following are equivalent*:

(a) $(X, \mathfrak{I})$ *is a perfectly normal* T_1*-space with a* σ*-point finite base.*

(b) $(X, \mathfrak{I})$ *is a normal metacompact Moore space.*

PROOF. (a)→(b) This follows from Bennett's result, that a perfectly normal T_1-space with a quasi-development is a Moore space, and Theorem 3. (b)→(a) This follows from Theorem 4 and the fact that a perfectly normal metacompact space is hereditarily metacompact.

THEOREM 6. *A topological space with a* σ*-point finite base has a* σ*-disjoint base iff it is hereditarily screenable.*

PROOF. Theorems 3 and 4.

5. **Some examples.** Corson and Michael [**8**] have exhibited a space which is T_2, Lindelöf, and hereditarily paracompact with a σ-disjoint base which is not metrizable. Heath [**10**] has an example of a completely regular nonnormal Moore space with a σ-point finite base but not a σ-disjoint base. Miščenko [**13**] has an example of a hereditarily Lindelöf T_2-space that is not regular which has a point-countable base but does not have a σ-point finite base. For further discussion of these last two examples, see Aull [**3**]. We will modify another example of Miščenko [**13**] to obtain an example of a hereditarily paracompact T_2-space with a point-countable base that does not have a σ-point finite base.

EXAMPLE. We define a topology $(X, \mathfrak{I})$ as follows. Let α be an ordinal number. We denote by $R(\alpha)$ the set of all ordinal numbers $\beta<\alpha$. Let N be the set of all natural numbers. We denote by X_α the set of all mappings $x=x(\gamma)$, $\gamma<\alpha$, of the set $R(\alpha)$ into N (i.e., the set of all sequences of order type α whose elements are natural numbers: $\{x_1, x_2, \cdots, x_\gamma, \cdots\}$; $\gamma<\alpha$, $x_\gamma \in N$). We set $X=\bigcup_{1<\alpha<\Omega} X_\alpha$, where Ω is the first uncountable ordinal number. We shall call the ordinal number α the length of element $x\in X_\alpha$. We shall say that the element x is an extension of the element y if the length of $x=\alpha>\beta=$ length of y and, for all $\gamma<\beta$, we have $x(\gamma)=y(\gamma)$. Let the length of x be equal to α. We denote by $U_n(x)$ the set consisting of the point x and of all $y\in X$ that are extensions of x and such that $y(\alpha)\geqq n$. Then $\mathfrak{B} = \{U_n(x)\}_{n=1}^\infty$, $x\in X$, is a base for a topology $\mathfrak{I}$ on X. This follows from the fact that if $y\neq x$ and $y\in U_k(x)$, then $U_n(y)\subset U_k(x)$ for all n.

We establish a series of properties of the base $\mathfrak{B}$.

(1) If neither of two elements x and y is an extension of the other, then $U_n(x)\cap U_m(y)=\varnothing$ for all n and m.

(2) If x is an extension of y but $x \notin U_m(y)$, then $U_n(x) \cap U_m(y) = \emptyset$ for all n. If $x \in U_n(y)$, $U_m(x) \subset U_n(y)$ for all m.

We show that the base $\mathfrak{B}$ is point-countable. If $x \in U_k(y)$, then x is an extension of y. The set of all y such that x is an extension of y is countable. Then there are only countably many sets $U_k(y)$ such that $x \in U_k(y)$.

The argument for $(X, \mathfrak{J})$ being hereditarily strongly paracompact is very similar to that for the original example of Miščenko being strongly paracompact.

We will show $(X, \mathfrak{J})$ does not have a σ-point finite base. Let $\mathfrak{U}$ be a base for $(X, \mathfrak{J})$ such that $\mathfrak{U} \subset \mathfrak{B}$. To deny that $\mathfrak{U}$ is σ-point finite, it will be sufficient to show that $\mathfrak{U}$ has a subfamily which is an uncountable, descending chain. We use transfinite induction; given an ordinal number α and $x \in X$ such that x is of length α and such that for any predecessor of x $(y < x)$ there exist $U_{n(y)}(y)$ such that the $\{U_{n(y)}(y)\}$ form a descending chain. If α is a limit ordinal, $x \in U_{n(y)}(y)$ for every $y < x$ and by (2), for each m, $U_m(x) \subset U_{n(y)}(y)$ for every $y < x$. If α is a nonlimit ordinal, then x has an immediate predecessor p and there exists z of length α such that $z \in U_{n(p)}(p)$. Then, by (2), $U_m(z) \subset U_{n(y)}(y)$ for every $y < x$. Let $\mathfrak{V}$ be any base for $(X, \mathfrak{J})$. There exists another base which is a subfamily of $\mathfrak{V}$, $\mathfrak{W}$ such that if $W \in \mathfrak{W}$, there exists $x \in X$ and n, $k \in I$ such that $U_n(x) \subset W \subset U_p(x)$. Furthermore, each W can be associated with only one x in the above manner. Let $\mathfrak{U}$ consist of all $U_n(x)$ such that $U_n(x) \subset W \subset U_p(x)$ for $W \in \mathfrak{W}$. Let $\mathfrak{S}$ be an uncountable descending chain of members of $\mathfrak{U}$. For each $S \in \mathfrak{S}$ there exists a distinct $W \in \mathfrak{W}$ such that $S \subset W$. The family $\mathfrak{W}$ is then not σ-finite, and since $\mathfrak{V}$ is an arbitrary base, $(X, \mathfrak{J})$ does not have a σ-point finite base.

6. **Some concluding remarks.** The question of the metrizability of the normal Moore space has an interesting history. See Jones [**11**] and, for some more recent developments, see the doctoral thesis of F. D. Tall [**17**].

In regard to many of the theorems proved in this paper there is the question of whether they can be proved with weaker conditions. For instance, in regard to Corollary 2A, are collectionwise normal, perfectly normal T_1-spaces with a point-countable base metrizable? In regard to Theorem 2 and the example of Heath [**10**] of a completely regular nonnormal space with a σ-point finite base but not a σ-disjoint base, can one construct a T_4-space with a σ-point finite base that does not have a σ-disjoint base? Such a space if perfectly normal would be a metacompact normal Moore space. Heath [**10**] has proved that if every metacompact normal Moore space is metrizable then every separable normal Moore space is metrizable. But J. H. Silver has proved that the existence of a nonmetrizable normal separable Moore space is consistent with Zermelo-Fraenkel set theory. For the proof, see F. D. Tall [**17**, p. 74].

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