NETS OF SUBGROUPS AND AMENABILITY

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ABSTRACT. We first investigate some convergence properties of nets in the topological space of closed subgroups of a locally compact group. We then use these properties to answer some questions concerning the amenable subgroups.

Introduction. Let G be a locally compact group and $\mathcal{K}(G)$ the set of all closed subgroups of G equipped with the compact-open topology (see §1 for the definition). If H is a closed (resp. closed normal) subgroup of G , then there is a canonical correspondence between the elements of $\mathcal{K}(G)$ contained in (resp. containing) H and the elements of $\mathcal{K}(H)$ (resp. $\mathcal{K}(G/H)$). In §2 we show these are homeomorphisms. We then give necessary and sufficient conditions for a net in $\mathcal{K}(G)$ directed by set containment (resp. inclusion) to converge to G (resp. the trivial subgroup).

In §3 we consider the collection of groups in $\mathcal{K}(G)$ which are amenable. We give sufficient conditions for the limit of a net of amenable subgroups to be amenable and another condition which is equivalent to amenability of G . Finally, if G is not amenable, then it does not contain a largest amenable subgroup; however, it does contain a maximal one.

1. Preliminaries. Recall [5, p. 427] that a basic open subset in the compact-open topology of $\mathcal{K}(G)$ is of the form $\mathfrak{U}(C, \mathfrak{F}) = \{K \in \mathcal{K}(G) : K \cap C = \emptyset, K \cap B \neq \emptyset, \text{ for each } B \in \mathfrak{F}\}$, where C is a compact subset of G and $\mathfrak{F}$ is a finite family of nonempty open subsets of G . (Note that C and $\mathfrak{F}$ may be empty.) With this topology, $\mathcal{K}(G)$ is a compact Hausdorff space. Let $\mathcal{S}(G)$ denote the space of subgroup representations of G , i.e. pairs (K, T) where $K \in \mathcal{K}(G)$ and T is a (unitary equivalence class of a) unitary representation of K . We assume $\mathcal{S}(G)$ to be equipped with its usual inner hull-kernel topology [5, p. 249]. E will denote the trivial subgroup $\{e\}$ of G and 1 its identity representation (it is also the regular representation). Let R (resp. I) denote the regular (resp. identity) representation of G . If $\{K_i\}$ is a net in $\mathcal{K}(G)$, I_i (resp. R_i) will denote the identity (resp. regular) representation of K_i . If $H, K \in \mathcal{K}(G)$ with $K \supseteq H$ and T a representa-

Received by the editors July 20, 1970.

AMS 1970 subject classifications. Primary 22D05; Secondary 22D10, 22D30.

Key words and phrases. Locally compact group, space of closed subgroups, net of subgroups, amenable group, weak containment, subgroup representation, induced representation.

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tion of H , then ${}_K U^T$ will denote the usual induced representation of K (see [7, p. 312] or [2]). In particular, if $K = K_i$ as above (resp. $K = G$), then we write ${}_i U^T$ (resp. U^T). We will use K to denote a generic element of $\mathcal{K}(G)$ and J (resp. P) its identity (resp. regular) representation. We also assume the reader is familiar with the notions of amenability [6] and weak containment [3].

LEMMA 1. *Let the notation be as above.*

- (i) *If $K \in \mathcal{K}(G)$, then $U^P = R$.*
- (ii) *G is amenable if and only if I is weakly contained in R .*
- (iii) *If $K \in \mathcal{K}(G)$ is normal, $\pi: G \rightarrow G/K$ the canonical mapping and R' the regular representation of G/K , then $U^J = R'\pi$. Also, G/K is amenable if and only if I is weakly contained in U^J .*

PROOF. (i) Clearly, ${}_K U^1 = P$ and $U^1 = R$. Hence, by inducing in stages [2, Theorem 1], we have $U^P = U^1 = R$. (ii) In view of [6, Theorem 3.5.2] we need only verify the reverse implication. This is proved as in the separable case [4, p. 260] making use of Lemma 4.1 of [1]. (iii) The first statement is clear. For the second, observe that the first part of Lemma 5.2 of [4] is valid for nonseparable groups as well.

The next lemma provides a slight generalization of [6, Theorem 2.3.3].

LEMMA 2. *If $K \in \mathcal{K}(G)$, then G is amenable if and only if K is amenable and I is weakly contained in U^J .*

PROOF. If G is amenable, then K is amenable [6, Theorem 2.3.2]. Also, G acts amenably on G/K [7, Corollary 3.2]. Consequently, I is weakly contained in U^J by the proof of Theorem 5.1 of [7].

Conversely, if K is amenable, then J is weakly contained in P implies U^J is weakly contained in $U^P = R$, i.e. I is weakly contained in R [3, p. 375].

2. Nets of subgroups. If $H \in \mathcal{K}(G)$, then let $\mathcal{K}_c(G, H)$ (resp. $\mathcal{K}_c(G, H)$) denote the collection of K in $\mathcal{K}(G)$ which contain (resp. are contained in) H .

PROPOSITION 1. *$\mathcal{K}_c(G, H)$ is closed in $\mathcal{K}(G)$ and the identity mapping $\mathcal{K}(H) \rightarrow \mathcal{K}_c(G, H)$ is a homeomorphism.*

PROOF. Since $\mathcal{K}(H)$ and $\mathcal{K}(G)$ are compact Hausdorff spaces, it suffices to show this mapping is continuous. Let K in $\mathcal{K}(H)$ be the limit of a net $\{K_i\}$ in $\mathcal{K}(H)$. Let $\mathcal{U}(C, \mathfrak{F})$ be a basic open neighborhood of K in $\mathcal{K}(G)$. Then $C_H = C \cap H$ is a compact subset of H and

$\mathfrak{F}_H = \{B \cap H : B \in \mathfrak{F}\}$ is a finite family of nonempty relatively open subsets of H ; in fact, $\mathfrak{U}(C_H, \mathfrak{F}_H)$ is an open neighborhood of K in $\mathcal{K}(H)$. Hence, there exists i_0 such that $i \geq i_0$ implies $K_i \cap C \cap H = \emptyset$ and $K_i \cap B \cap H \neq \emptyset$, $B \in \mathfrak{F}$. This completes the proof since $K_i \cap H = K_i$, for each i .

PROPOSITION 2. *Let $H \in \mathcal{K}(G)$ be normal. Then $\mathcal{K}^c(G, H)$ is closed in $\mathcal{K}(G)$ and the canonical bijection $\mathcal{K}(G/H) \rightarrow \mathcal{K}^c(G, H)$ given by $K \rightarrow \pi^{-1}(K)$ (where $\pi: G \rightarrow G/H$ is the canonical map) is a homeomorphism.*

PROOF. As in the previous case, it suffices to show that the given mapping is continuous. Let K in $\mathcal{K}(G/H)$ be the limit of a net $\{K_i\}$ in $\mathcal{K}(G/H)$. Let $\mathfrak{U}(C, \mathfrak{F})$ be a basic open neighborhood of $\pi^{-1}(K)$ in $\mathcal{K}(G)$. Then $C' = \pi(C)$ is a compact subset of G/H and $\mathfrak{F}' = \{\pi(B) : B \in \mathfrak{F}\}$ is a finite family of nonempty open subsets of G/H ; in fact, it is easy to verify that $\mathfrak{U}(C', \mathfrak{F}')$ is an open neighborhood of K' in $\mathcal{K}(G/H)$. Hence, there exists i_0 such that $i \geq i_0$ implies $K'_i \cap C' = \emptyset$ and $K'_i \cap B' \neq \emptyset$, $B' \in \mathfrak{F}'$. It is also easy to verify that this in turn implies that $\pi^{-1}(K_i) \in \mathfrak{U}(C, \mathfrak{F})$, for $i \geq i_0$.

PROPOSITION 3. *If $H \in \mathcal{K}(G)$ is open, the mapping $K \rightarrow H \cap K$ of $\mathcal{K}(G)$ into itself is continuous.*

PROOF. Let $\{K_i\}$ be a net in $\mathcal{K}(G)$ converging to K in $\mathcal{K}(G)$. Let $\mathfrak{U}(C, \mathfrak{F})$ be a basic open neighborhood of $H \cap K$ in $\mathcal{K}(G)$ and let C_H and $\mathfrak{F}_H$ be defined as in the proof of Proposition 1. Then it is clear that $\mathfrak{U}(C_H, \mathfrak{F}_H)$ is an open neighborhood of K . Thus, there exists i_0 such that $i \geq i_0$ implies $K_i \in \mathfrak{U}(C_H, \mathfrak{F}_H)$, i.e. $H \cap K_i \in \mathfrak{U}(C, \mathfrak{F})$.

THEOREM 1. *Let $\{K_i\}$ be a net in $\mathcal{K}(G)$ directed by $\supseteq$, i.e. $i \geq j$ if and only if $K_i \supseteq K_j$. Then $K_i \rightarrow G$ if and only if $G = \text{cl}(\cup K_i)$.*

PROOF. Suppose $K_i \rightarrow G$ and $\text{cl}(\cup K_i) \neq G$. Then there exists x in G such that $x \notin \text{cl}(\cup K_i)$. Let V be an open neighborhood of x such that $V \cap \text{cl}(\cup K_i) = \emptyset$ (G is regular). Then $\mathfrak{U}(\emptyset, \{V\})$ is an open neighborhood of G ; but for each i , $K_i \cap V = \emptyset$, i.e. $K_i \notin \mathfrak{U}(\emptyset, \{V\})$, which is a contradiction.

Conversely, suppose $G = \text{cl}(\cup K_i)$. A basic open neighborhood of G is of the form $\mathfrak{U}(\emptyset, \mathfrak{F})$, where $\mathfrak{F}$ is arbitrary. Let $\mathfrak{F} = \{B_1, \dots, B_n\}$. Since $\cup K_i$ is dense in G , for each $1 \leq k \leq n$, there exists i_k such that $B_k \cap K_{i_k} \neq \emptyset$. By hypothesis, there exists i_0 such that $i_0 \geq i_k$, $1 \leq k \leq n$, i.e. $K_{i_0} \supseteq \cup \{K_{i_k} : 1 \leq k \leq n\}$. Now suppose $i \geq i_0$, i.e. $K_i \supseteq K_{i_0}$. Then for each $1 \leq k \leq n$, we have

$$K_i \cap B_k \supseteq \bigcup \{K_{i_j} \cap B_j : 1 \leq j \leq n\} \supseteq K_{i_k} \cap B_k \neq \emptyset.$$

Thus, if $i \geq i_0$, then $K_i \in \mathcal{U}(\emptyset, \mathcal{F})$, i.e. $K_i \rightarrow G$.

COROLLARY. *If $K \in \mathcal{K}(G)$ and $K_i \subseteq K$ for all i , then $K_i \rightarrow K$ if and only if $K = \text{cl}(\bigcup K_i)$.*

PROOF. Apply Proposition 1.

THEOREM 2. *Let $\{K_i : i \in A\}$ be a net in $\mathcal{K}(G)$ directed by $\subseteq$ i.e. $i \geq j$ if and only if $K_i \subseteq K_j$. Then $K_i \rightarrow E$ if and only if $E = \bigcap K_i$.*

PROOF. Suppose $K_i \rightarrow E$. We may assume $G \neq E$. If $E \neq \bigcap K_i$, then there exists $x \in \bigcap K_i$ such that $x \neq e$. Hence, $\mathcal{U}(\{x\}, \emptyset)$ is an open neighborhood of E such that for each $i \in A$, $x \in K_i$, i.e. $K_i \notin \mathcal{U}(\{x\}, \emptyset)$, which is a contradiction.

Conversely, suppose $E = \bigcap K_i$ and let $\mathcal{U}(C, \mathcal{F})$ be a basic open neighborhood of E . Then $e \in C$ and $e \in B$ for each $B \in \mathcal{F}$. If the net $\{K_i : i \in A\}$ is not eventually in $\mathcal{U}(C, \mathcal{F})$, then for each $i \in A$, there exists $i' \in A$ such that $i' \geq i$ and $K_{i'} \cap C \neq \emptyset$. Let $A' = \{i' : i' \in A\}$. Then A' is a directed subset of A and $\{K_{i'} : i' \in A'\}$ is a subnet of $\{K_i : i \in A\}$ satisfying $K_{i'} \cap C \neq \emptyset$, for each i' . Let $x_{i'} \in K_{i'} \cap C$. Then $\{x_{i'} : i' \in A'\}$ is a net in C . Since C is compact, there exists a subnet $\{y_j : j \in A''\}$ and $x_0 \in C$ such that $y_j \rightarrow x_0$. Let $b : A'' \rightarrow A'$ be the mapping defining this subnet. Now define $H_j = K_{b(j)}$, $j \in A''$. Hence, $y_j = x_{b(j)}$ is in H_j for each $j \in A''$. It is easy to verify that $\{H_j : j \in A''\}$ is a subnet of $\{K_i : i \in A\}$ with defining map $a \circ b : A'' \rightarrow A$, where $a : A' \rightarrow A$ is the injection. We now show that $x_0 \in \bigcap \{H_j : j \in A''\}$.

Fix arbitrary $j_0 \in A''$. Then $b(j_0) \in A'$ and there exists $j_1 \in A''$ such that $j \geq j_1$ in A'' implies $b(j) \geq b(j_0)$ in A' . But by assumption, this is equivalent to $K_{b(j)} \subseteq K_{b(j_0)}$, i.e. $H_j \subseteq H_{j_0}$. Let $A'_1 = \{j \in A'' : j \geq j_1\}$. Then the net $\{y_j : j \in A'_1\}$ also converges to x_0 . Furthermore, $j \geq j_1$ implies $y_j \in H_j \subseteq H_{j_0}$. Thus, x_0 is an accumulation point of H_{j_0} , i.e. $x_0 \in H_{j_0}$. Since j_0 was arbitrary, we have $x_0 \in \bigcap \{H_j : j \in A''\}$.

Now let i be an arbitrary element of A . Then there exists $j \in A''$ such that $a \circ b(j) \geq i$ in A , i.e. $H_j \subseteq K_i$. Therefore, $x_0 \in K_i$ which implies $x_0 = e$ by our hypothesis. This is a contradiction since $C \cap E = \emptyset$ and the proof is complete.

COROLLARY. *If $K \in \mathcal{K}(G)$ is normal and $K_i \supseteq K$ for each i , then $K_i \rightarrow K$ if and only if $K = \bigcap K_i$.*

PROOF. Apply Proposition 2.

3. Amenability. The previous results can be used to answer some

questions on amenable subgroups. Let $\mathcal{K}_a(G)$ denote the set of amenable groups in $\mathcal{K}(G)$. We will need the following:

LEMMA 3. *Let $\{K_i\}$ be a net in $\mathcal{K}(G)$ and K an element of $\mathcal{K}(G)$ such that $K_i \subseteq K$ for all i . The following are equivalent:*

- (i) $K_i \rightarrow K$ in $\mathcal{K}(G)$.
- (ii) $(K_i, I_i) \rightarrow (K, J)$ in $\mathcal{S}(G)$.
- (iii) $(K_i, R_i) \rightarrow (K, P)$ in $\mathcal{S}(G)$.

PROOF. In view of Lemma 2.5 of [5], we need only verify that (i) implies (ii) and (iii). The net $\{(K_i, K, J)\}$ converges to (K, K, J) in $\mathcal{K}(G) \times \mathcal{S}(G)$. Since restricting is continuous [5, Theorem 3.2], we see that $(K_i, J|K_i) \rightarrow (K, J)$ which gives (ii) since $J|K_i = I_i$. Similarly, the net $\{(K_i, E, 1)\}$ converges to $(K, E, 1)$ in $\mathcal{K}(G) \times \mathcal{S}(G)$. Since inducing is continuous [5, Theorem 4.2], we have $(K_i, {}_iU^1) \rightarrow (K, {}_K U^1)$, which gives (iii) since $R_i = {}_iU^1$ and $P = {}_K U^1$.

LEMMA 4. *Let the notation be as in Lemma 3, but now assume that $K_i \supseteq K$ for all i . Then (i), (ii) and (iii) of Lemma 3 are again equivalent.*

PROOF. As before, we need only show that (i) implies (ii) and (iii). In this case, $(K_i, G, I) \rightarrow (K, G, I)$, so that $(K_i, I_i) = (K_i, I|K_i) \rightarrow (K, I|K) = (K, J)$. Similarly, $(K_i, K, P) \rightarrow (K, K, P)$, so that $(K_i, R_i) = (K_i, {}_iU^P) \rightarrow (K, P)$.

THEOREM 3. *Let $\{K_i\}$ be a net in $\mathcal{K}_a(G)$ and K an element of $\mathcal{K}(G)$ such that $K \supseteq K_i$, for each i , and $K_i \rightarrow K$. Then $K \in \mathcal{K}_a(G)$.*

PROOF. By Lemma 3(ii), we have $(K_i, I_i) \rightarrow (K, J)$, which implies $(K, K_i, I_i) \rightarrow (K, K, J)$ in $\mathcal{K}(G) \times \mathcal{S}(G)$. Hence, $(K, {}_K U^{I_i}) \rightarrow (K, J)$ [5, Theorem 4.2], i.e. J is weakly contained in $\{{}_K U^{I_i}\}$ as representations of K (see Proposition 1.2 and Lemma 2.6 of [5]). However, since each K_i is amenable, we also know that each I_i is weakly contained in R_i . Thus, ${}_K U^{I_i}$ is weakly contained in ${}_K U^{R_i} = P$ [5, Theorem 4.3], i.e. $\{{}_K U^{I_i}\}$ is weakly contained in P . Therefore, J is weakly contained in P [3, p. 375] and the proof is complete.

COROLLARY 1. *Suppose $K \in \mathcal{K}(G)$ is open and there exists a net $\{K_i\}$ in $\mathcal{K}_a(G)$ converging to K . Then $K \in \mathcal{K}_a(G)$. In particular, if G is discrete, then $\mathcal{K}_a(G)$ is closed in $\mathcal{K}(G)$.*

PROOF. By Proposition 3 we have $K_i \cap K \rightarrow K$, where $K_i \cap K \subseteq K$ and $K_i \cap K \in \mathcal{K}_a(G)$ [6, Theorem 2.3.2]. Now apply the theorem.

Observe now that Theorem 2.3.4 of [6] is a particular case of Theorem 3.

COROLLARY 2. *If G is a limit of a net of amenable subgroups, then G is amenable. In particular, this is the case if G is a directed union of such groups.*

PROOF. Apply Theorem 1.

REMARK. If G is nontrivial, then $\mathcal{K}_a(G)$ must be nontrivial. If $x \in G$ and $x \neq e$, then the closed subgroup generated by x will be abelian and thus a nontrivial element of $\mathcal{K}_a(G)$.

THEOREM 4. *G is amenable if and only if the partially ordered set $(\mathcal{K}_a(G), \supseteq)$ is directed.*

PROOF. The forward implication is trivial. Suppose $(\mathcal{K}_a(G), \supseteq)$ is directed. Then $\bigcup \{H : H \in \mathcal{K}_a(G)\}$ is a subgroup of G whose closure K belongs to $\mathcal{K}_a(G)$ (apply the Corollary to Theorem 1 and Theorem 3). Since the conjugates of K must also belong to $\mathcal{K}_a(G)$, it follows that K must be normal. In fact, K must be G itself. Otherwise, G/K contains a nontrivial amenable subgroup whose inverse image is an element of $\mathcal{K}_a(G)$ (Lemmas 1 and 2) which is strictly larger than K .

This theorem shows that a nonamenable group cannot contain a largest amenable subgroup. However, the next best thing is true.

THEOREM 5. *The partially ordered set $(\mathcal{K}_a(G), \supseteq)$ contains a maximal element.*

PROOF. Let $\mathcal{L}$ be a linearly ordered subset of $\mathcal{K}_a(G)$. Then $\mathcal{L}$ is a net directed by $\supseteq$. Hence, it must converge to $\text{cl}(\bigcup \{L : L \in \mathcal{L}\}) = H$ which is amenable (Corollary to Theorem 1 and Theorem 3). H is clearly an upper bound for $\mathcal{L}$. Now apply Zorn's lemma.

As an illustration of this theorem, let G be the free group on two generators a and b . Let G_a (resp. G_b) denote the subgroup generated by a (resp. b). Then G_a and G_b are amenable since they are abelian. We now show that each is a maximal amenable subgroup of G (recall that G is not amenable).

Let H be an amenable subgroup of G such that $H \supseteq G_a$ and $H \neq G_a$. Then H is also free. Let n be the cardinality of a set of generators for H . If $n \geq 2$, then H contains a copy of G and cannot be amenable. Thus, $n = 1$. Let $c \in G$ be a generator for H . Since $a \in H$, there exists an integer q such that $a = c^q$. Since q cannot be 0 or 1, we have a non-trivial relation in G which is a contradiction.

CONCLUDING REMARKS. It is interesting to observe that the main results of this section, Theorems 3, 4 and 5, are consequences (primarily) of Theorem 1 and Lemma 3. Since Theorem 2 and Lemma 4 are natural analogues of these last two results in the opposite direc-

tion, it seems likely that for a suitable replacement of $\mathcal{K}_a(G)$, Theorems 3, 4 and 5 also have natural analogues in the opposite direction. We conjecture that $\mathcal{K}_1(G) = \{K \in \mathcal{K}(G) : I \text{ is weakly contained in } U^J\}$ is this replacement (see [5, p. 442]).

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